

'02 Institute for Cosmic Ray Research

11/22

Neutrino Oscillation Probability and CP Violation in Matter

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Reference

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Phys. Lett. B544 (2002) 286
2. K. Kimura, A. T. and H. Yokomakura
Phys. Rev. D66 (2002) 073005
3. K. Kimura, A. T. and H. Yokomakura
Phys. Lett. B537 (2002) 86
4. H. Yokomakura, K. Kimura, and A. T.
Phys. Lett. B496 (2000) 175

Matter Invariant Approach

$$\tilde{H} = H + \frac{1}{2E} \begin{pmatrix} ii & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

$$\implies \tilde{H}_{\alpha\beta} = H_{\alpha\beta} \quad (\text{except for } H_{ee})$$

We can make matter invariant identities by using this relation.

$$\text{Im}(\tilde{H}_{e\mu}\tilde{H}_{\mu\tau}\tilde{H}_{\tau e}) = \text{Im}(H_{e\mu}H_{\mu\tau}H_{\tau e})$$



$$\tilde{H} = \tilde{U} \text{diag}(\lambda_1, \lambda_2, \lambda_3) \tilde{U}^\dagger$$

$$\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}\tilde{J} = \Delta_{12}\Delta_{23}\Delta_{31}J$$

(Naumov-Harrison-Scott identity)

$$\implies \tilde{J} = \text{Im}(\tilde{U}\tilde{U}^*\tilde{U}^*\tilde{U}) = \frac{\Delta_{12}\Delta_{23}\Delta_{31}}{\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} J$$

$$J = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin \delta$$

Matter Effects	\implies	Δ_{ij}
CP Effects	\implies	$\sin \delta$

Both matter effects and CP effects are simple.
Can we also derive the expression of $\text{Re}\tilde{J}$?

Introduction

Present

- Atmospheric Neutrino

$$\theta_{23} \sim \frac{\pi}{4}, \quad |\Delta_{31}| \sim 3 \times 10^{-3} \text{ eV}^2$$

- Solar Neutrino

$$\theta_{12} \sim \frac{\pi}{4}, \quad |\Delta_{21}| \sim 1 \times 10^{-4} \text{ eV}^2$$

- CHOOZ

$$\theta_{13} \leq 0.2$$



$$J = \underbrace{s_{12}c_{12}}_{\text{large}} \underbrace{s_{23}c_{23}}_{\text{large}} s_{13}c_{13}^2 \sin \delta$$

Future

- Superbeam or Neutrino Factory

$$\theta_{13}, \quad \text{sgn}(\Delta_{31}), \quad \delta$$

Our Main Purpose

Measurement of CP phase

Formula of Oscillation Probability

$\nu_e \rightarrow \nu_\mu$ transition

$$P(\nu_e \rightarrow \nu_\mu) = | < \nu_\mu | U e^{-iHt} U^\dagger | \nu_e > |^2$$

\downarrow
 $U_{\alpha i}$: MNS matrix

$$U^\dagger H U = \frac{1}{2E} \text{diag}(0, \Delta_{21}, \Delta_{31})$$

$$P(\nu_e \rightarrow \nu_\mu) = -4 \underbrace{\sum_{(ij)} \text{cyclic} \text{Re} J_{e\mu}^{ij} \sin^2 \Delta'_{ij}}_{\text{CP even}} - 2 \underbrace{\sum_{(ij)} \text{cyclic} J \sin 2\Delta'_{ij}}_{\text{CP odd}}$$

$$J_{e\mu}^{ij} \equiv U_{ei} U_{\mu i}^* (U_{ej} U_{\mu j}^*)^* \quad J \equiv \text{Im } J_{e\mu}^{12} \quad \Delta'_{ij} \equiv \frac{L}{4E} \Delta_{ij}$$

vacuum \Rightarrow matter

$$H \Rightarrow \tilde{H} = H + \frac{1}{2E} \begin{pmatrix} & 0 \\ 0 & \end{pmatrix} \quad \tilde{U}^\dagger \tilde{H} \tilde{U} = \frac{1}{2E} \text{diag}(\lambda_1, \lambda_2, \lambda_3)$$

Probability in matter is obtained by the replacements

$$\begin{cases} U_{\alpha i} \rightarrow \tilde{U}_{\alpha i} \\ \Delta_{ij} \rightarrow \tilde{\Delta}_{ij} \equiv \lambda_i - \lambda_j \end{cases}$$

Approach to CP phase and Problem

CP violation

$$\begin{aligned} P_{CP} &= P(\nu_e \rightarrow \nu_\mu; \delta) - P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu; -\delta) \\ &\stackrel{\text{cyclic}}{=} -4 \sum_{(ij)} J \sin 2\Delta'_{ij} \propto \sin \delta \end{aligned}$$

$P_{CP} \neq 0 \Rightarrow$ CP violation (Vacuum)

$$\begin{aligned} P_{CP} &= P(\nu_e \rightarrow \nu_\mu; \delta, a) - P(\bar{\nu}_e \rightarrow \bar{\nu}_\mu; -\delta, -a) \\ &\stackrel{\text{cyclic}}{\neq} -4 \sum_{(ij)} \tilde{J} \sin 2\tilde{\Delta}'_{ij} \end{aligned}$$

$P_{CP} \neq 0 \Rightarrow$ CP violation or Matter Effects (Matter)

* We focus to the exact understanding of pure CP effects and Matter effects in $P(\nu_e \rightarrow \nu_\mu)$.

Assumptions

- Constant Matter
- Standard Three Active Neutrino Scenario.

Effective Mixing Angles and CP Phase

Effective Mixing Angles and CP Phase

$$\begin{aligned}\sin \tilde{\theta}_{13}^2 &= \frac{\lambda_3^2 - \alpha\lambda_3 + \beta}{\tilde{\Delta}_{13}\tilde{\Delta}_{23}} \\ \sin \tilde{\theta}_{12}^2 &= \frac{-(\lambda_2^2 - \alpha\lambda_2 + \beta)\tilde{\Delta}_{31}}{(\lambda_1^2 - \alpha\lambda_1 + \beta)\tilde{\Delta}_{32} - (\lambda_2^2 - \alpha\lambda_2 + \beta)\tilde{\Delta}_{31}} \\ \sin \tilde{\theta}_{23}^2 &= \frac{G^2 s_{23}^2 + F^2 c_{23}^2 + 2GFs_{23}c_{23}\cos\delta}{G^2 + F^2} \\ e^{-i\tilde{\delta}} &= \frac{(G^2 e^{-i\delta} - F^2 e^{i\delta})s_{23}c_{23} + GF(c_{23}^2 - s_{23}^2)}{\sqrt{(G^2 s_{23}^2 + F^2 c_{23}^2 + 2GFs_{23}c_{23}\cos\delta)(G^2 c_{23}^2 + F^2 s_{23}^2 - 2GFs_{23}c_{23}\cos\delta)}}\end{aligned}$$

Where

$$\begin{aligned}\alpha &= m_3^2 c_{13}^2 + m_2^2 (c_{13}^2 c_{12}^2 + s_{13}^2) + m_1^2 (c_{13}^2 s_{12}^2 + s_{13}^2) \\ \beta &= m_3^2 c_{13}^2 (m_2^2 c_{13}^2 + m_1^2 s_{12}^2) + m_2^2 m_1^2 s_{13}^2 \\ Gs_{23} &= [\Delta_{31}(\lambda_3 - m_1^2 - \Delta_{21}) - \Delta_{21}(\lambda_3 - m_1^2 - \Delta_{31})s_{12}^2]c_{13}s_{13} \\ Fc_{23} &= (\lambda_3 - m_1^2 - \Delta_{31})\Delta_{21}c_{12}s_{12}c_{13}\end{aligned}$$

H.W.Zaglauer and K.H.Schwarzer, Z.Phys.C40,273 (1988)

$\tilde{\theta}_{23}$ and $\tilde{\delta}$ are especially complicated.

$$\tilde{J} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{23} \sin 2\tilde{\theta}_{31} \cos \tilde{\theta}_{31} \sin \tilde{\delta} = ?$$

Simple Derivation of Exact Formula

$$P(\nu_e \rightarrow \nu_\mu) = -4 \sum_{(ij)}^{\text{cyclic}} \operatorname{Re} \tilde{J}_{e\mu}^{ij} \sin^2 \tilde{\Delta}'_{ij} - 2 \sum_{(ij)}^{\text{cyclic}} \tilde{J} \sin 2\tilde{\Delta}'_{ij}$$

$$\tilde{J}_{e\mu}^{ij} = \tilde{U}_{ei} \tilde{U}_{\mu i}^* (\tilde{U}_{ej} \tilde{U}_{\mu j}^*)^* \quad \tilde{J} = \operatorname{Im} \tilde{J}_{e\mu}^{12}$$

We only have to calculate $\tilde{U}_{ei} \tilde{U}_{\mu i}^*$ to learn $P(\nu_e \rightarrow \nu_\mu)$.

Unitarity

$$\tilde{U}_{e1} \tilde{U}_{\mu 1}^* + \tilde{U}_{e2} \tilde{U}_{\mu 2}^* + \tilde{U}_{e3} \tilde{U}_{\mu 3}^* = 0$$

$$\tilde{H}_{e\mu} = H_{e\mu}$$

$$\lambda_1 \tilde{U}_{e1} \tilde{U}_{\mu 1}^* + \lambda_2 \tilde{U}_{e2} \tilde{U}_{\mu 2}^* + \lambda_3 \tilde{U}_{e3} \tilde{U}_{\mu 3}^* = p$$

$$\tilde{\mathcal{H}}_{e\mu} = \mathcal{H}_{e\mu}$$

$$\lambda_2 \lambda_3 \tilde{U}_{e1} \tilde{U}_{\mu 1}^* + \lambda_3 \lambda_1 \tilde{U}_{e2} \tilde{U}_{\mu 2}^* + \lambda_1 \lambda_2 \tilde{U}_{e3} \tilde{U}_{\mu 3}^* = q$$

$$(\mathcal{H}_{e\mu} \equiv H_{e\tau} H_{\tau\mu} - H_{e\mu} H_{\tau\tau})$$

$$\Rightarrow \tilde{U}_{ei} \tilde{U}_{\mu i}^* = \frac{p + q}{\tilde{\Delta}'_{ij} \tilde{\Delta}'_{ij}} \quad \begin{cases} p = 2E H_{e\mu} \\ q = (2E)^2 \mathcal{H}_{e\mu} \end{cases}$$

Matter Effect $\implies \lambda_i$

CP Effects $\implies \lambda_i, p, q$

Effective Masses

$$\begin{aligned}\lambda_1 &= \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t} \left[u + \sqrt{3(1-u^2)} \right] \\ \lambda_2 &= \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t} \left[u - \sqrt{3(1-u^2)} \right] \\ \lambda_3 &= \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2 - 3t}\end{aligned}$$

$$s = \Delta_{21} + \Delta_{31} + \dots$$

$$t = \Delta_{21}\Delta_{31} + [\Delta_{21}(1-s_{12}^2c_{13}^2) + \Delta_{31}(1-s_{13}^2)]$$

$$u = \cos \left[\frac{1}{3} \cos^{-1} \left(\frac{2s^3 - 9st + 27\Delta_{21}\Delta_{31}c_{12}^2c_{13}^2}{2(s^2 - 3t)^{3/2}} \right) \right]$$

Remark

There is no CP dependence in effective masses.

Matter Invariant Quantities p and q

$$\begin{cases} p = \Delta_{21} U_{e2} U_{\mu 2}^* + \Delta_{31} U_{e3} U_{\mu 3}^* \\ q = \Delta_{21} \Delta_{31} U_{e1} U_{\mu 1}^* \end{cases}$$

* Standard Parametrization

$$U = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

$$p = p_1 e^{-i\delta} + p_2, \quad q = q_1 e^{-i\delta} + q_2$$

↓

$$\tilde{U}_{ei} \tilde{U}_{\mu i}^* = \frac{p\lambda_i + q}{\tilde{\Delta}_{ji} \tilde{\Delta}_{ki}}$$

$$\boxed{\tilde{U}_{ei} \tilde{U}_{\mu i}^* = X_1 e^{-i\delta} + X_2}$$

CP dependence of $P(\nu_e \rightarrow \nu_\mu)$

$$P(\nu_e \rightarrow \nu_\mu) = A \cos \delta + B \sin \delta + C$$

$P(\nu_e \rightarrow \nu_\mu)$ has only linear terms in $\cos \delta$ and $\sin \delta$.

Exact

$$A = \sum_{(ijk)}^{\text{cyclic}} \frac{-8J_r \Delta_{21} [\Delta_{31} \lambda_k (\lambda_k - \Delta_{31}) + A_k]}{\tilde{\Delta}_{jk}^2 \tilde{\Delta}_{ki}^2} \cos \tilde{\Delta}'_{ij} \sin \tilde{\Delta}'_{jk} \sin \tilde{\Delta}'_{ki}$$

$$B = \frac{8\Delta_{12}\Delta_{23}\Delta_{31}}{\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} J_r \sin \tilde{\Delta}'_{12} \sin \tilde{\Delta}'_{23} \sin \tilde{\Delta}'_{31}$$

$$C = \sum_{(ij)}^{\text{cyclic}} \frac{-4[s_{13}^2 s_{23}^2 c_{13}^2 \Delta_{31}^2 \lambda_i \lambda_j + C_{ij}]}{\tilde{\Delta}_{ij} \tilde{\Delta}_{12} \tilde{\Delta}_{23} \tilde{\Delta}_{31}} \sin^2 \tilde{\Delta}'_{ij}$$



$$\lambda_1 \sim \Delta_{21}, \lambda_2 \sim \Delta_{31}, \lambda_3 \sim a$$

$$(\lambda_1 \sim \Delta_{21}, \lambda_2 \sim a, \lambda_3 \sim \Delta_{31})$$

Approximate

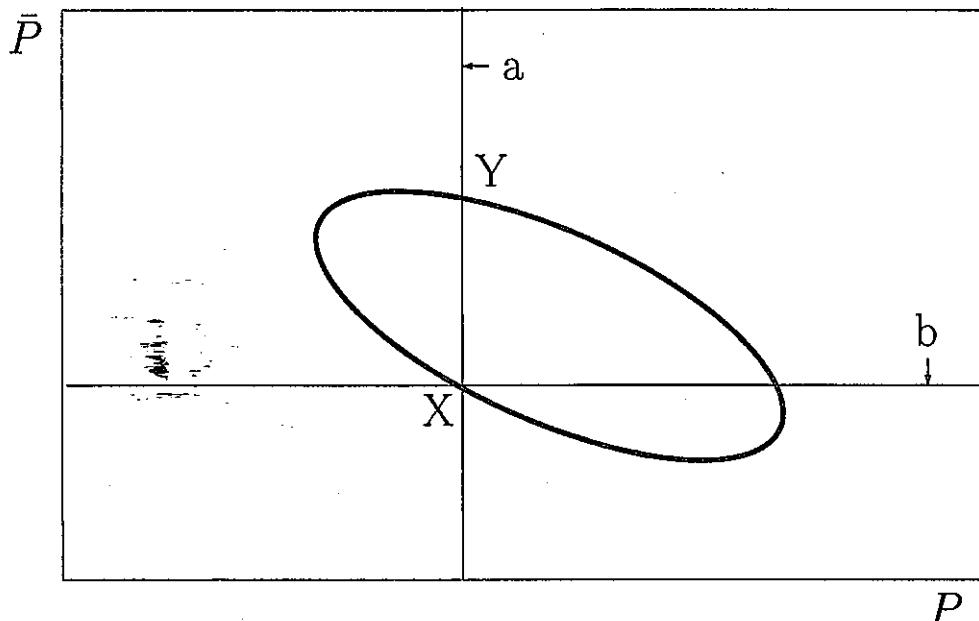
$$A \simeq \frac{8J_r \Delta_{21} \Delta_{31}}{a(\Delta_{31} - a)} \cos \Delta'_{31} \sin a' \sin(\Delta_{31} - a)'$$

$$B \simeq \frac{8J_r \Delta_{21} \Delta_{31}}{a(\Delta_{31} - a)} \sin \Delta'_{31} \sin a' \sin(\Delta_{31} - a)'$$

$$C \simeq \frac{4\Delta_{31}^2 s_{23}^2 s_{13}^2 c_{13}^2}{-(\Delta_{31} - a)^2} \sin^2(\Delta_{31} - a)'$$

CP Trajectory as an Important Tool

CP trajectory



Minakata and Nunokawa, hep-ph/0108085

$$P = A \cos \delta + B \sin \delta + C$$

$$\bar{P} = \bar{A} \cos \delta + \bar{B} \sin \delta + \bar{C}$$

CP phase can be determined by

$$\sin \delta = \frac{(\bar{A}P - A\bar{P}) - (\bar{A}C - A\bar{C})}{\bar{A}B - A\bar{B}}$$

$$\cos \delta = \frac{(\bar{B}P - B\bar{P}) - (\bar{B}C - B\bar{C})}{\bar{B}A - B\bar{A}}$$

8. Generalization to Another Channels

$$\tilde{H} = H + \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow = 91\% 15$

- $\tilde{U}_{\alpha 1} \tilde{U}_{\beta 1}^* + \tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* + \tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* = \delta_{\alpha \beta}$

$$\tilde{H}_{\alpha \beta} = \tilde{P}_{\alpha \beta} \quad 15$$

- $\lambda_1 \tilde{U}_{\alpha 1} \tilde{U}_{\beta 1}^* + \lambda_2 \tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* + \lambda_3 \tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* = \tilde{P}_{\alpha \beta}$

$$\tilde{\mathcal{H}}_{\alpha \beta}^* = \tilde{H}_{\alpha \tau} \tilde{H}_{\beta \tau} - \tilde{H}_{\alpha \beta} \tilde{H}_{\tau \tau} = \tilde{q}_{\alpha \beta} \quad 15$$

- $\lambda_1 \lambda_2 \tilde{U}_{\alpha 1} \tilde{U}_{\beta 1}^* + \lambda_2 \lambda_3 \tilde{U}_{\alpha 2} \tilde{U}_{\beta 2}^* + \lambda_1 \lambda_3 \tilde{U}_{\alpha 3} \tilde{U}_{\beta 3}^* = \tilde{q}_{\alpha \beta}$

\Downarrow 連立方程式をとく

$$\tilde{U}_{\alpha i} \tilde{U}_{\beta i}^* = \frac{\tilde{P}_{\alpha \beta} \lambda_i + \tilde{q}_{\alpha \beta} - \delta_{\alpha \beta} \lambda_i (\lambda_j + \lambda_k)}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik}}$$

$$= \frac{\delta_{\alpha \beta} \lambda_i^2 + (\tilde{P}_{\alpha \beta} - \delta_{\alpha \beta} \text{tr } \tilde{H}) \lambda_i + \tilde{q}_{\alpha \beta}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik}}$$

$$\text{tr } H = \lambda_1 + \lambda_2 + \lambda_3$$

15

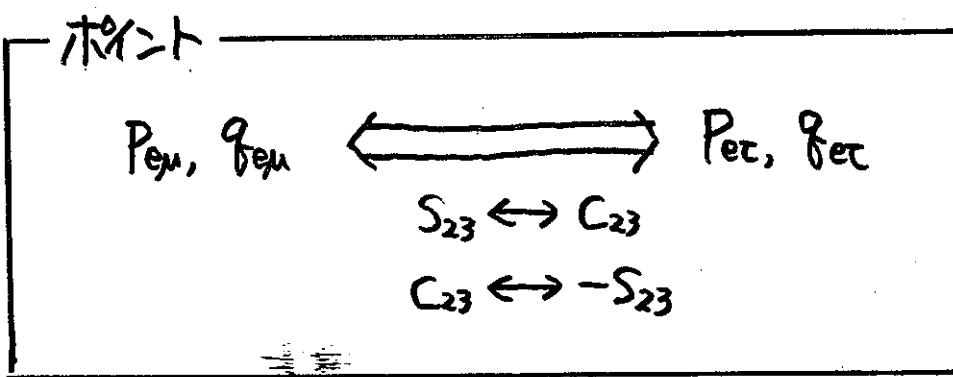
$$P(\nu_e \rightarrow \nu_\tau) = -4 \sum_{ij}^{\text{cyclic}} \operatorname{Re} \tilde{T}_{e\tau}^{ij} \sin^2 \left(\frac{\tilde{\Delta}_{ij} L}{4E} \right) + 2 \tilde{T} \sum_{ij}^{\text{cyclic}} \sin \left(\frac{\tilde{\Delta}_{ij} L}{2E} \right)$$

$$\tilde{U}_{ei} \tilde{U}_{\tau i}^* = \frac{P_{e\tau} \lambda_i + Q_{e\tau}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ie}}$$

$$\begin{cases} P_{e\tau} = \Delta_{21} U_{e2} U_{\tau 2}^* + \Delta_{31} U_{e3} U_{\tau 3}^* = (P_{e\tau})_1 e^{-i\delta} + (P_{e\tau})_2 \\ Q_{e\tau} = \Delta_{21} \Delta_{31} U_{e1} U_{\tau 1}^* = (Q_{e\tau})_1 e^{-i\delta} + (Q_{e\tau})_2 \end{cases}$$

$$\therefore (P_{e\tau})_1 = (\Delta_{31} - \Delta_{21} S_{12}^2) C_{23} S_{13} C_{13}, (P_{e\tau})_2 = -\Delta_{21} S_{12} C_{12} S_{23} C_{13}$$

$$(Q_{e\tau})_1 = -\Delta_{31} \Delta_{21} C_{12}^2 C_{23} S_{13} C_{13}, (Q_{e\tau})_2 = \Delta_{31} \Delta_{21} S_{12} C_{12} S_{23} C_{13}$$



CP 1族対称性は

$$P(\nu_e \rightarrow \nu_\tau) = A_{e\tau} \cos \delta - B_{e\tau} \sin \delta + C_{e\tau}$$

$$P(\nu_\mu \rightarrow \nu_e) = -4 \sum_{ij}^{\text{cyclic}} \text{Re} \tilde{J}_{\mu e} \tilde{\sim}_{ij} \sin^2 \left(\frac{\tilde{\Delta}_{ij} L}{4E} \right) - 2 \tilde{J} \sum_{ij}^{\text{cyclic}} \sin \left(\frac{\tilde{\Delta}_{ij} L}{2E} \right)$$

$$\tilde{U}_{\mu i} \tilde{U}_{e i}^* = \frac{P_{\mu e} \lambda_i + \tilde{q}_{\mu e}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ie}}$$

$$\begin{aligned} \tilde{q}_{\mu e} &= \tilde{H}_{\mu e}^* = \tilde{H}_{e\tau} \tilde{H}_{\mu e} - \tilde{H}_{e\tau} \tilde{H}_{\mu \tau} \\ &= \tilde{H}_{e\tau} \tilde{H}_{\mu e} - (\tilde{H}_{e\tau} + a) \tilde{H}_{\mu \tau} \\ &= \frac{\tilde{H}_{e\tau} \tilde{H}_{\mu e} - \tilde{H}_{e\tau} \tilde{H}_{\mu \tau}}{\tilde{q}_{\mu e}} - a \frac{\tilde{H}_{\mu \tau}}{P_{\mu e}} \end{aligned}$$

$$\tilde{U}_{\mu i} \tilde{U}_{e i}^* = \frac{P_{\mu e} (\lambda_i - a) + \tilde{q}_{\mu e}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ie}}$$

$$\begin{cases} P_{\mu e} = \Delta_{21} U_{\mu 2} U_{e 2}^* + \Delta_{31} U_{\mu 3} U_{e 3}^* = (P_{\mu e})_1 e^{-i\delta} + (P_{\mu e})_2 + (P_{\mu e})_3 e^{i\delta} \\ q_{\mu e} = \Delta_{21} \Delta_{31} U_{\mu 1} U_{e 1}^* = (q_{\mu e})_1 e^{-i\delta} + (q_{\mu e})_2 + (q_{\mu e})_3 e^{i\delta} \end{cases}$$

$$\therefore (P_{\mu e})_1 = -\Delta_{21} S_{12} C_{12} C_{23}^2 S_{13}, \quad (P_{\mu e})_2 = [\Delta_{31} C_{13}^2 - \Delta_{21} (C_{12}^2 - S_{12}^2 S_{13}^2)] S_{23} C_{23}$$

$$(P_{\mu e})_3 = \Delta_{21} S_{12} C_{12} S_{23}^2 S_{13}$$

$$(q_{\mu e})_1 = \Delta_{31} \Delta_{21} S_{12} C_{12} C_{23}^2 S_{13}, \quad (q_{\mu e})_2 = \Delta_{31} \Delta_{21} (-S_{12}^2 + C_{12}^2 S_{13}^2) S_{23} C_{23}$$

$$(q_{\mu e})_3 = -\Delta_{31} \Delta_{21} S_{12} C_{12} S_{23}^2 S_{13}$$

CP 亂れの性質

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} \cos \delta + A' \cos 2\delta + B \sin \delta + C_{\mu e}$$

$$\begin{aligned}
 P(\nu_e \rightarrow \nu_e) &= 1 - P(\nu_e \rightarrow \nu_\mu) - P(\nu_e \rightarrow \nu_\tau) \\
 &= 1 + 4 \sum_{ij}^{\text{cyclic}} (\underbrace{\text{Re } \tilde{J}_{e\mu}^{ij} + \text{Re } \tilde{J}_{e\tau}^{ij}}_{\Delta_{ij} \tilde{\Delta}_{12} \tilde{\Delta}_{23} \tilde{\Delta}_{31}}) \sin^2 \left(\frac{\tilde{\Delta}_{ij} L}{4E} \right) \\
 &\quad \cdot \frac{(|P_{e\mu}|^2 + |P_{e\tau}|^2) \lambda_i \lambda_j + (|q_{e\mu}|^2 + |q_{e\tau}|^2) + [\text{Re}(P_{e\mu} q_{e\mu}^* + P_{e\tau} q_{e\tau}^*)] (\lambda_i \lambda_j)}{\Delta_{ij}}
 \end{aligned}$$

$$\left. \begin{aligned}
 |P|^2 &= P_1^2 + P_2^2 + 2P_1 P_2 \cos \delta \\
 |q_B|^2 &= q_{f1}^2 + q_{f2}^2 + 2q_{f1} q_{f2} \cos \delta \\
 \text{Re}(Pq^*) &= P_1 q_{f1} + P_2 q_{f2} + (P_1 q_{f2} + P_2 q_{f1}) \cos \delta
 \end{aligned} \right)$$

$$\begin{aligned}
 (P_{e\mu})_1 &= (\Delta_{31} - \Delta_{21} S_{12}^2) S_{23} S_{13} C_{13} & (P_{e\tau})_1 &= (\Delta_{31} - \Delta_{21} S_{12}^2) C_{23} S_{13} C_{13} \\
 (P_{e\mu})_2 &= \Delta_{21} C_{12} C_{23} C_{13} & (P_{e\tau})_2 &= -\Delta_{21} C_{12} S_{23} C_{13} \\
 (q_{e\mu})_1 &= -\Delta_{31} \Delta_{21} C_{12}^2 S_{23} S_{13} C_{13} & (q_{e\tau})_1 &= -\Delta_{31} \Delta_{21} C_{12}^2 C_{23} S_{13} C_{13} \\
 (q_{e\mu})_2 &= -\Delta_{31} \Delta_{21} S_{12} C_{12} S_{23} C_{13} & (q_{e\tau})_2 &= +\Delta_{31} \Delta_{21} S_{12} C_{12} S_{23} C_{13}
 \end{aligned}$$

$$\begin{aligned}
 \text{Re } \tilde{J}_{e\mu}^{ij} &= S_{23} C_{23} \cos \delta \times \boxed{\square} + \triangle \\
 \text{Re } \tilde{J}_{e\tau}^{ij} &= -S_{23} C_{23} \cos \delta \times \boxed{\square} + \circ
 \end{aligned}$$

よって CP 係数性は

$$\begin{aligned}
 P(\nu_e \rightarrow \nu_e) &= \underline{\text{Cee}} \\
 &\quad \text{S1には全く存在しない}
 \end{aligned}$$

$$P(\nu_\mu \rightarrow \nu_\mu) = 1 - P(\nu_\mu \rightarrow \nu_e) - P(\nu_\mu \rightarrow \nu_\tau)$$

$$P(\nu_\mu \rightarrow \nu_e) = A_{\mu e} \cos \delta \quad -B \sin \delta + C_{\mu e}$$

$$+) \quad P(\nu_\mu \rightarrow \nu_\tau) = A_{\mu \tau} \cos \delta + A' \cos 2\delta + B \sin \delta + C_{\mu \tau}$$

$$(A_{\mu e} + A_{\mu \tau}) \cos \delta + A' \cos 2\delta \quad + (C_{\mu e} + C_{\mu \tau})$$

$\cancel{d \rightarrow ?}$

$$\underline{P(\nu_\mu \rightarrow \nu_\mu) = A_{\mu \mu} \cos \delta - A' \cos 2\delta + C_{\mu \mu}}$$

$$P(\nu_\tau \rightarrow \nu_\tau) = 1 - P(\nu_e \rightarrow \nu_\tau) - P(\nu_\mu \rightarrow \nu_\tau)$$

$$P(\nu_e \rightarrow \nu_\tau) = A_{e \tau} \cos \delta \quad -B \sin \delta + C_{e \tau}$$

$$+) \quad P(\nu_\mu \rightarrow \nu_\tau) = A_{\mu \tau} \cos \delta + A' \cos 2\delta + B \sin \delta + C_{\mu \tau}$$

$$(A_{e \tau} + A_{\mu \tau}) \cos \delta + A' \cos 2\delta \quad + (C_{e \tau} + C_{\mu \tau})$$

$\cancel{d \rightarrow ?}$

$$\underline{P(\nu_\tau \rightarrow \nu_\tau) = A_{\tau \tau} \cos \delta - A' \cos 2\delta + C_{\tau \tau}}$$

9. Effective Mixing Angles and CP phase

$$S_{12}^{m^2} = \frac{-(\lambda_2^2 - \alpha\lambda_2 + \beta)\tilde{\Delta}_{31}}{\tilde{\Delta}_{32}(\lambda_1^2 - \alpha\lambda_1 + \beta) - \tilde{\Delta}_{31}(\lambda_2^2 - \alpha\lambda_2 + \beta)}$$

$$S_{13}^{m^2} = \frac{\lambda_3^2 - \alpha\lambda_3 + \beta}{\tilde{\Delta}_{31}\tilde{\Delta}_{32}}$$

$$S_{23}^{m^2} = \frac{E^2 S_{23}^2 + F^2 C_{23}^2 + 2EF S_{23} C_{23} C_8}{E^2 + F^2}$$

$$e^{-i\delta^m} = \frac{(E^2 e^{-i\delta} - F^2 e^{i\delta}) S_{23} C_{23} + EF(C_{23}^2 - S_{23}^2)}{\sqrt{(E^2 S_{23}^2 + F^2 C_{23}^2 + 2EF S_{23} C_{23} C_8)(E^2 C_{23}^2 + F^2 S_{23}^2 - 2EF S_{23} C_{23} C_8)}}$$

$\approx 2^\circ$

$$\alpha = m_3^2 C_{13}^2 + m_2^2 (C_{13}^2 C_{12}^2 + S_{13}^2) + m_1^2 (C_{13}^2 S_{12}^2 + S_{13}^2)$$

$$\beta = m_3^2 C_{13}^2 (m_2^2 C_{12}^2 + m_1^2 S_{12}^2) + m_2^2 m_1^2 S_{13}^2$$

$$E = [\Delta_{31}(\lambda_3 - m_1^2 - \Delta_{21}) - \Delta_{21}(\lambda_3 - m_1^2 - \Delta_{31}) S_{12}^2] C_{13} S_{13}$$

$$F = (\lambda_3 - m_1^2 - \Delta_{31}) \Delta_{21} C_{12} S_{12} C_{13}$$

$$X_{123} (c_{12} c_{13} - s_{12} s_{13}) (c_{12} c_{13} - 2s_{12} s_{13})$$

$$\tilde{U}_{\alpha i} \tilde{U}_{\beta i}^* = \frac{\delta_{\alpha\beta} \lambda_i^2 + (\tilde{P}_{\alpha\beta} - \delta_{\alpha\beta} \text{tr} \tilde{H}) \lambda_i + \tilde{q}_{\alpha\beta}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik}}$$

$\alpha = \beta = e \neq 3$

$$|\tilde{U}_{ei}|^2 = \frac{\lambda_i^2 - (P_{mu} + P_{ee}) \lambda_i + q_{ee}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik}}$$

$$S_{13}^{m2} = |\tilde{U}_{e3}|^2 = \frac{\lambda_3^2 - (P_{mu} + P_{ee}) \lambda_3 + q_{ee}}{\tilde{\Delta}_{31} \tilde{\Delta}_{32}} \quad \text{OK}$$

$$|\tilde{U}_{e2}|^2 = S_{12}^{m2} C_{13}^{m2}, \quad |\tilde{U}_{e1}|^2 = C_{12}^{m2} C_{13}^{m2} \quad \text{d) } \quad t_{12}^{m2} = \frac{|\tilde{U}_{e2}|^2}{|\tilde{U}_{e1}|^2}$$

$$S_{12}^{m2} = \frac{t_{12}^{m2}}{1 + t_{12}^{m2}} = \frac{-(\lambda_2^2 - \alpha \lambda_2 + \beta) \tilde{\Delta}_{31}}{\tilde{\Delta}_{32}(\lambda_1^2 - \alpha \lambda_1 + \beta) - \tilde{\Delta}_{31}(\lambda_2^2 - \alpha \lambda_2 + \beta)} \quad \text{OK}$$

$$|\tilde{U}_{\mu i}|^2 = \frac{|\tilde{U}_{ei} \tilde{U}_{\mu i}^*|^2}{|\tilde{U}_{ei}|^2} = \frac{|P_{mu} \lambda_i + q_{\mu e}|^2}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik} [\lambda_i^2 - \alpha \lambda_i + \beta]}$$

$$|\tilde{U}_{\tau i}|^2 = \frac{|\tilde{U}_{ei} \tilde{U}_{\tau i}^*|^2}{|\tilde{U}_{ei}|^2} = \frac{|P_{\tau e} \lambda_i + q_{\tau e}|^2}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik} [\lambda_i^2 - \alpha \lambda_i + \beta]}$$

$$|\tilde{U}_{\mu 3}|^2 = S_{23}^{m2} C_{13}^{m2}, \quad |\tilde{U}_{\tau 3}|^2 = C_{23}^{m2} C_{13}^{m2} \quad \text{d) } \quad t_{23}^{m2} = \frac{|\tilde{U}_{\mu 3}|^2}{|\tilde{U}_{\tau 3}|^2}$$

→?

$$S_{23}^{m2} = \frac{t_{23}^{m2}}{1 + t_{23}^{m2}} = \frac{(P_{\text{eu}}\lambda_i + Q_{\text{eu}})^2}{|P_{\text{eu}}\lambda_i + Q_{\text{eu}}|^2 + |P_{\text{ec}}\lambda_i + Q_{\text{ec}}|^2}$$
$$= \frac{E^2 S_{23}^2 + F^2 C_{23}^2 + 2EF S_{23} C_{23} C_S}{E^2 + F^2} \quad \text{OK}$$

$$C_{23}^{m2} = 1 - S_{23}^{m2} = \frac{E^2 C_{23}^2 + F^2 S_{23}^2 - 2EF S_{23} C_{23} C_S}{E^2 + F^2}$$

$$\begin{aligned} \|\tilde{U}_{\mu 1}\|^2 &= | -C_{23}^m S_{12}^m - S_{23}^m S_{13}^m C_{12}^m e^{i\delta_m}|^2 (= 1 + \lambda) ? \\ &\text{or} \\ &\text{Toshev relation} \\ &\sin 2\theta_{23}^m \sin \delta^m = \sin 2\theta_{23} \sin \delta \quad \text{E(R)} ? \end{aligned}$$

$$e^{-i\delta_m} = \frac{(E^2 e^{-i\delta} - F^2 e^{i\delta}) S_{23} C_{23} + EF (C_{23}^2 - S_{23}^2)}{\sqrt{(E^2 S_{23}^2 + F^2 C_{23}^2 + 2EF S_{23} C_{23} C_S)(E^2 C_{23}^2 + F^2 S_{23}^2 - 2EF S_{23} C_{23} C_S)}}$$

固有値方程式

$$\begin{pmatrix} H_{ee} + \alpha - \lambda & H_{em} & H_{ec} \\ H_{me} & H_{mm} - \lambda & H_{mc} \\ H_{ce} & H_{cm} & H_{cc} - \lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} H_{ee} - \lambda & H_{em} & H_{ec} \\ H_{me} & H_{mm} - \lambda & H_{mc} \\ H_{ce} & H_{cm} & H_{cc} - \lambda \end{pmatrix} + \alpha \begin{pmatrix} H_{mm} - \lambda & H_{mc} \\ H_{cm} & H_{cc} - \lambda \end{pmatrix} = 0$$

$$-(\lambda - m_1^2)(\lambda - m_2^2)(\lambda - m_3^2) + \alpha(\lambda - k_1)(\lambda - k_2) = 0 \quad \dots (*)$$

$h = \begin{pmatrix} H_{mm} & H_{mc} \\ H_{cm} & H_{cc} \end{pmatrix}$ の固有値

- 方

$$|\tilde{U}_{ei}|^2 = \frac{\lambda_i^2 - (P_{\text{out}} + P_{\text{cc}})\lambda_i + q_{\text{free}}}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik}} \quad \det h = k_1 k_2$$

$$= \frac{(\lambda_i - k_1)(\lambda_i - k_2)}{\tilde{\Delta}_{ij} \tilde{\Delta}_{ik}}$$

$$|\tilde{U}_{e1}|^2 |\tilde{U}_{e2}|^2 |\tilde{U}_{e3}|^2 = \frac{\prod_{i=1,2} (\lambda_i - k_{ei})(\lambda_i - k_{ci})(\lambda_i - k_{si})}{(\tilde{\Delta}_{12} \tilde{\Delta}_{23} \tilde{\Delta}_{31})^2} \quad (*) \text{ を用} \rightarrow$$

$$= \frac{\prod_{i=1,2} (m_i^2 - k_{ei})(m_i^2 - k_{ci})(m_i^2 - k_{si})}{(\tilde{\Delta}_{12} \tilde{\Delta}_{23} \tilde{\Delta}_{31})^2}$$

$$\tilde{\Delta}_{12} \tilde{\Delta}_{23} \tilde{\Delta}_{31} |\tilde{U}_{e1} \tilde{U}_{e2} \tilde{U}_{e3}| = \Delta_{12} \Delta_{23} \Delta_{31} \frac{|U_{e1} U_{e2} U_{e3}|}{S_{12} C_{12} S_{13} C_{13}^2}$$

$$\tilde{\Delta}_{12} \tilde{\Delta}_{23} \tilde{\Delta}_{31} S_{12}^m S_{13}^{m^2} C_{13}^m S_{23}^m C_{23}^m S_3^m = \Delta_{12} \Delta_{23} \Delta_{31} S_{12} C_{12} S_{13} C_{13}^2 S_{23} C_{23} S_3$$

(Harrison-Scott) (Toshev)

2つの恒等式に分離される

$P(V_e \rightarrow V_\mu)$ の CP 依存性を Zaglauer らの結果からみると

$$\text{Re } J_{e\mu}^{12} = - (C_2^2 - S_{12}^2) J_r \cos \delta + S_{12}^2 C_2^2 C_{13}^2 (S_{23}^2 S_{13}^2 - C_{23}^2)$$

$$\text{Re } J_{e\mu}^{23} = J_r \cos \delta - S_{12}^2 S_{23}^2 C_{13}^2 S_{13}^2$$

$$\text{Re } J_{e\mu}^{31} = - J_r \cos \delta - C_{12}^2 S_{23}^2 C_{13}^2 S_{13}^2$$

∴ $J_r = S_{12} C_{12} S_{23} C_{23} S_{13} C_{13}^2$

- $\cos \delta$ は $S_{23} C_{23}$ とセットで現れています !!
CP 依存性がカントンになります
- $S_{23}^{m^2}$ や $C_{23}^{m^2}$ は $\cos \delta, \sin \delta$ の 1 次の項のみ !!

Matter Invariant Approach と同じ結果が得られます。

Decomposition of NHS Identity

- Toshev identity

$$\tilde{s}_{23}\tilde{c}_{23}s_{\delta} = s_{23}c_{23}s_{\delta}$$

Toshev identity only depends on θ_{23} and δ

- New identity

$$\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}\tilde{s}_{12}\tilde{c}_{12}\tilde{s}_{13}\tilde{c}_{13}^2 = \Delta_{12}\Delta_{23}\Delta_{31}s_{12}c_{12}s_{13}c_{13}^2$$

New identity only depends on θ_{12} and θ_{13}

- Naumov-Harrison-Scott identity

$$\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}\tilde{s}_{12}\tilde{c}_{12}\tilde{s}_{13}\tilde{c}_{13}^2\tilde{s}_{23}\tilde{c}_{23}s_{\delta} = \Delta_{12}\Delta_{23}\Delta_{31}s_{12}c_{12}s_{13}c_{13}^2s_{23}c_{23}s_{\delta}$$

NHS identity is decomposed by the product of
 θ_{12} θ_{13} and θ_{23} δ

Question:

Can we extend this fact to the case of arbitrary matter?

CP dependence of Hamiltonian

Effective Hamiltonian

$$H = \begin{pmatrix} H_{ee} & H_{e\mu} & H_{e\tau} \\ H_{\mu e} & H_{\mu\mu} & H_{\mu\tau} \\ H_{\tau e} & H_{\tau\mu} & H_{\tau\tau} \end{pmatrix}$$

Matrix Elements

$$\begin{aligned}
 H_{ee} &= \Delta_{21} c_{13}^2 s_{12}^2 + \Delta_{31} s_{13}^2 + a(x) \\
 H_{\mu\mu} &= \Delta_{21} (c_{23}^2 c_{12}^2 + s_{23}^2 s_{13}^2 s_{12}^2 - 2 c_{23} s_{23} c_{12} s_{12} s_{13} \cos \delta) \\
 &\quad + \Delta_{31} s_{23}^2 c_{13}^2 \\
 H_{\tau\tau} &= \Delta_{21} (s_{23}^2 c_{12}^2 + c_{23}^2 s_{13}^2 s_{12}^2 + 2 c_{23} s_{23} c_{12} s_{12} s_{13} \cos \delta) \\
 &\quad + \Delta_{31} c_{23}^2 c_{13}^2 \\
 H_{e\mu} &= \Delta_{21} (c_{13} c_{23} s_{12} c_{12} - c_{13} s_{13} s_{23} s_{12}^2 e^{-i\delta}) + \Delta_{31} s_{13} c_{13} s_{23} e^{i\delta} \\
 H_{\mu\tau} &= \Delta_{21} (s_{23} c_{23} (s_{12}^2 - c_{12}^2) + s_{13} s_{12} c_{12} (s_{23}^2 - c_{23}^2) e^{i\delta}) \\
 &\quad + \Delta_{31} s_{23} c_{23} c_{13}^2 \\
 H_{\tau e} &= \Delta_{21} (c_{13} s_{23} s_{12} c_{12} + c_{13} s_{13} c_{23} s_{12}^2 e^{i\delta}) + \Delta_{31} s_{13} c_{13} c_{23} e^{-i\delta}
 \end{aligned}$$

Remarkable Facts

- H_{ee} only depends on matter potential $a(x)$
- H_{ee} does not depend on δ and θ_{23}

We guess

δ and θ_{23} would be not influented by Matter effect

even if it is in arbitrary matter profile case.

Decomposition of Hamiltonian

Standard Parametrization

$$U = O_{23}\Gamma_\delta O_{13}\Gamma_\delta^\dagger O_{12}$$

where

$$\Gamma_\delta = \text{diag}(1, 1, e^{i\delta})$$

Decomposition of Hamiltonian

$$\begin{aligned} H &= \frac{1}{2E}(U \text{diag}(0, \Delta_{21}, \Delta_{31}) U^\dagger + \text{diag}[a(t), 0, 0]) \\ &= \frac{1}{2E} O_{23} \Gamma_\delta (O_{13} O_{12} \text{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T \\ &\quad + \text{diag}[a(t), 0, 0]) \Gamma_\delta^\dagger O_{23}^T \end{aligned}$$

where we use

$$\begin{aligned} [O_{23}, \text{diag}[a(t), 0, 0]] &= 0 \\ [\Gamma_\delta, \text{diag}[a(t), 0, 0]] &= 0 \\ [\Gamma_\delta, \text{diag}(0, \Delta_{21}, \Delta_{31})] &= 0 \end{aligned}$$

and we define

$$H'(t) = \frac{1}{2E} O_{13} O_{12} \text{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T$$

Decomposition of Hamiltonian

$$H(t) = [O_{23} \Gamma_\delta] H'(t) [O_{23} \Gamma_\delta]^\dagger$$

Separation of θ_{12}, θ_{13} parts and θ_{23}, δ parts

CP Dependence in Arbitrary Matter

Schrödinger equation

$$i \frac{\partial \nu}{\partial t} = H(t) \nu, \quad i \frac{\partial \nu'}{\partial t} = H'(t) \nu'$$

where

$$\nu' = [O_{23} \Gamma_\delta]^\dagger \nu,$$

Time evolution operator $S(t)$

$$\nu(t) = S(t) \nu(0), \quad \nu'(t) = S'(t) \nu'(0)$$

and

$$S(t) = [O_{23} \Gamma_\delta] S'(t) [O_{23} \Gamma_\delta]^\dagger$$

$$S_{ee} = S'_{ee}$$

$$S_{\mu e} = S'_{\mu e} c_{23} + S'_{\tau e} s_{23} e^{i\delta}$$

$$S_{\tau e} = -S'_{\mu e} s_{23} + S'_{\tau e} c_{23} e^{i\delta}$$

$$S_{\mu\mu} = S'_{\mu\mu} c_{23}^2 + S'_{\mu\tau} c_{23} s_{23} e^{-i\delta} + S'_{\tau\mu} c_{23} s_{23} e^{i\delta} + S'_{\tau\tau} s_{23}^2$$

$$S_{\tau\mu} = -S'_{\mu\mu} c_{23} s_{23} - S'_{\mu\tau} s_{23}^2 e^{-i\delta} + S'_{\tau\mu} c_{23}^2 e^{i\delta} + S'_{\tau\tau} c_{23} s_{23}$$

$$S_{\tau\tau} = S'_{\mu\mu} s_{23}^2 - S'_{\mu\tau} c_{23} s_{23} e^{-i\delta} - S'_{\tau\mu} c_{23} s_{23} e^{i\delta} + S'_{\tau\tau} c_{23}^2$$

Main Results $(P(\nu_\alpha \rightarrow \nu_\beta) = P_{\alpha\beta})$

$$P_{ee} = C_{ee}$$

$$P_{e\mu} = A_{e\mu} \cos \delta + B_{e\mu} \sin \delta + C_{e\mu}$$

$$P_{e\tau} = A_{e\tau} \cos \delta + B_{e\tau} \sin \delta + C_{e\tau}$$

$$P_{\mu\mu} = A_{\mu\mu} \cos \delta + B_{\mu\mu} \sin \delta + C_{\mu\mu} + D_{\mu\mu} \cos 2\delta + E_{\mu\mu} \sin 2\delta$$

$$P_{\tau\tau} = A_{\tau\tau} \cos \delta + B_{\tau\tau} \sin \delta + C_{\tau\tau} + D_{\tau\tau} \cos 2\delta + E_{\tau\tau} \sin 2\delta$$

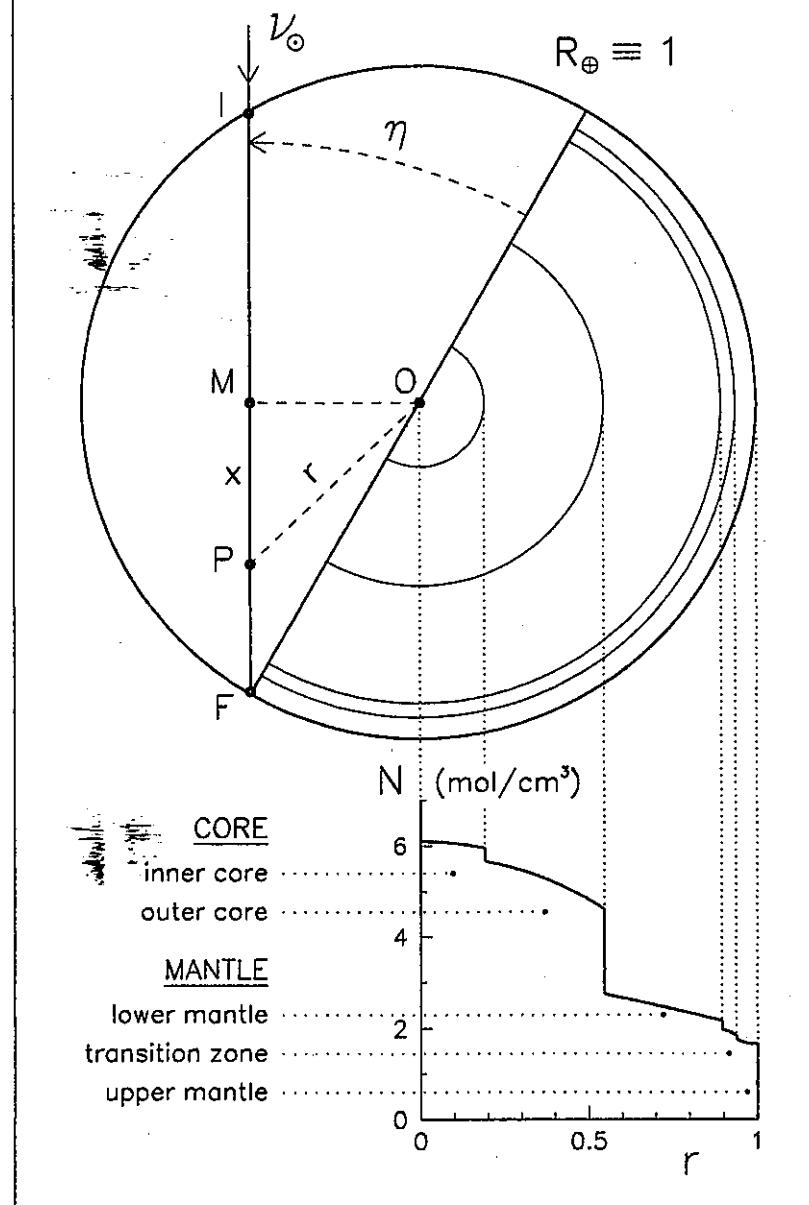
$$P_{\mu\tau} = A_{\mu\tau} \cos \delta + B_{\mu\tau} \sin \delta + C_{\mu\tau} + D_{\mu\tau} \cos 2\delta + E_{\mu\tau} \sin 2\delta$$

Overall Feature of CP Dependence

- * $P_{ee} = C_{ee} = |S'_{ee}|^2,$
- * $P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu},$
 $A_{e\mu} = 2\operatorname{Re}[S'^*_{\mu e} S'_{\tau e}] c_{23} s_{23},$
 $B_{e\mu} = -2\operatorname{Im}[S'^*_{\mu e} S'_{\tau e}] c_{23} s_{23},$
 $C_{e\mu} = |S'_{\mu e}|^2 c_{23}^2 + |S'_{\tau e}|^2 s_{23}^2,$
- * $P_{e\tau} = A_{e\tau}\cos\delta + B_{e\tau}\sin\delta + C_{e\tau},$
 $A_{e\tau} = -2\operatorname{Re}[S'^*_{\mu e} S'_{\tau e}] c_{23} s_{23},$
 $B_{e\tau} = 2\operatorname{Im}[S'^*_{\mu e} S'_{\tau e}] c_{23} s_{23},$
 $C_{e\tau} = |S'_{\mu e}|^2 c_{23}^2 + |S'_{\tau e}|^2 c_{23}^2,$
- * $P_{\mu\mu} = A_{\mu\mu}\cos\delta + B_{\mu\mu}\sin\delta + C_{\mu\mu} + D_{\mu\mu}\cos 2\delta + E_{\mu\mu}\sin 2\delta,$
 $A_{\mu\mu} = 2\operatorname{Re}[(S'_{\mu\mu} c_{23}^2 + S'_{\tau\tau} s_{23}^2)^*(S'_{\tau\mu} + S'_{\mu\tau})] c_{23} s_{23},$
 $B_{\mu\mu} = -2\operatorname{Im}[(S'_{\mu\mu} c_{23}^2 + S'_{\tau\tau} s_{23}^2)^*(S'_{\tau\mu} - S'_{\mu\tau})] c_{23} s_{23},$
 $C_{\mu\mu} = |S'_{\mu\mu} c_{23}^2 + S'_{\tau\tau} s_{23}^2|^2 + (|S'_{\mu\tau}|^2 + |S'_{\tau\mu}|^2) c_{23}^2 s_{23}^2$
 $D_{\mu\mu} = 2\operatorname{Re}[S'^*_{\tau\mu} S'_{\mu\tau}] c_{23}^2 s_{23}^2,$
 $E_{\mu\mu} = 2\operatorname{Im}[S'^*_{\tau\mu} S'_{\mu\tau}] c_{23}^2 s_{23}^2,$
- * $P_{\tau\tau} = A_{\tau\tau}\cos\delta + B_{\tau\tau}\sin\delta + C_{\tau\tau} + D_{\tau\tau}\cos 2\delta + E_{\tau\tau}\sin 2\delta,$
 $A_{\tau\tau} = -2\operatorname{Re}[(S'_{\mu\mu} s_{23}^2 + S'_{\tau\tau} c_{23}^2)^*(S'_{\tau\mu} + S'_{\mu\tau})] c_{23} s_{23},$
 $B_{\tau\tau} = 2\operatorname{Im}[(S'_{\mu\mu} s_{23}^2 + S'_{\tau\tau} c_{23}^2)^*(S'_{\tau\mu} - S'_{\mu\tau})] c_{23} s_{23},$
 $C_{\tau\tau} = |S'_{\mu\mu} s_{23}^2 + S'_{\tau\tau} c_{23}^2|^2 + (|S'_{\mu\tau}|^2 + |S'_{\tau\mu}|^2) c_{23}^2 s_{23}^2$
 $D_{\tau\tau} = 2\operatorname{Re}[S'^*_{\tau\mu} S'_{\mu\tau}] c_{23}^2 s_{23}^2,$
 $E_{\tau\tau} = 2\operatorname{Im}[S'^*_{\tau\mu} S'_{\mu\tau}] c_{23}^2 s_{23}^2,$
- * $P_{\mu\tau} = A_{\mu\tau}\cos\delta + B_{\mu\tau}\sin\delta + C_{\mu\tau} + D_{\mu\tau}\cos 2\delta + E_{\mu\tau}\sin 2\delta,$
 $A_{\mu\tau} = -2\operatorname{Re}[(S'_{\mu\mu} - S'_{\tau\tau})^*(S'_{\tau\mu} c_{23}^2 - S'_{\mu\tau} s_{23}^2)] c_{23} s_{23},$
 $B_{\mu\tau} = 2\operatorname{Im}[(S'_{\mu\mu} - S'_{\tau\tau})^*(S'_{\tau\mu} c_{23}^2 + S'_{\mu\tau} s_{23}^2)] c_{23} s_{23},$
 $C_{\mu\tau} = |S'_{\mu\tau} s_{23}^2 - S'_{\tau\mu} c_{23}^2|^2 + (|S'_{\mu\mu}|^2 + |S'_{\tau\tau}|^2) c_{23}^2 s_{23}^2$
 $D_{\mu\tau} = -2\operatorname{Re}[S'^*_{\tau\mu} S'_{\mu\tau}] c_{23}^2 s_{23}^2,$
 $E_{\mu\tau} = -2\operatorname{Im}[S'^*_{\tau\mu} S'_{\mu\tau}] c_{23}^2 s_{23}^2,$

Preliminary Reference Earth Model

I = ν entry point
F = ν endpoint (detector)
M = trajectory midpoint
P = generic ν position
 x = MP = trajectory coordinate
 r = OP = radial distance
 η = nadir angle

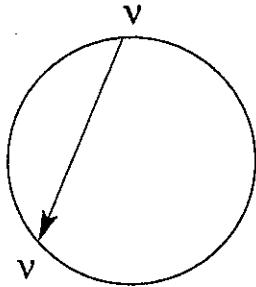


CP Dependence in Symmetric Matter

PREM is 1D model, so matter potential is symmetric.

Time evolution operator is symmetric

$$S'_{\alpha\beta} = S'_{\beta\alpha}$$



$$\begin{aligned} (S')^T &= (S_1 S_2 S_3 \cdots S_3 S_2 S_1)^T \\ &= S_1^T S_2^T S_3^T \cdots S_3^T S_2^T S_1^T \\ &= S_1 S_2 S_3 \cdots S_3 S_2 S_1 = S' \end{aligned}$$

In symmetric matter

$$\begin{aligned} B_{\mu\mu} &= -2c_{23}s_{23}\text{Im}[(S'_{\mu\mu}c_{23}^2 + S'_{\tau\tau}s_{23}^2)^*(S'_{\tau\mu} - S'_{\mu\tau})] = 0 \\ E_{\mu\mu} &= 2\text{Im}[S'_{\tau\mu}^* S'_{\mu\tau}]c_{23}^2 s_{23}^2 = 2\text{Im}[|S'_{\mu\tau}|^2]c_{23}^2 s_{23}^2 = 0 \\ B_{\tau\tau} &= 2c_{23}s_{23}\text{Im}[(S'_{\mu\mu}s_{23}^2 + S'_{\tau\tau}c_{23}^2)^*(S'_{\tau\mu} - S'_{\mu\tau})] = 0 \\ E_{\tau\tau} &= 2\text{Im}[S'_{\tau\mu}^* S'_{\mu\tau}]c_{23}^2 s_{23}^2 = 2\text{Im}[|S'_{\mu\tau}|^2]c_{23}^2 s_{23}^2 = 0 \\ E_{\mu\tau} &= -2\text{Im}[S'_{\tau\mu}^* S'_{\mu\tau}]c_{23}^2 s_{23}^2 = -2\text{Im}[|S'_{\mu\tau}|^2]c_{23}^2 s_{23}^2 = 0 \\ E_{\tau\mu} &= 2\text{Im}[S'_{\mu\tau}^* S'_{\tau\mu}]c_{23}^2 s_{23}^2 = 2\text{Im}[|S'_{\tau\mu}|^2]c_{23}^2 s_{23}^2 = 0 \end{aligned}$$

Main Result

$$\begin{aligned} P_{\mu\tau} &= A_{\mu\tau}\cos\delta + B_{\mu\tau}\sin\delta + C_{\mu\tau} + D_{\mu\tau}\cos 2\delta \\ P_{\mu\mu} &= A_{\mu\mu}\cos\delta + C_{\mu\mu} + D_{\mu\mu}\cos 2\delta \\ P_{\tau\tau} &= A_{\tau\tau}\cos\delta + C_{\tau\tau} + D_{\tau\tau}\cos 2\delta \end{aligned}$$

Comment on other Exact Formula

Question:

Why is ZS exact formula so complicated ?

Answer:

$$\begin{aligned}
 H &= \frac{1}{2E} O_{23} \Gamma_\delta (O_{13} O_{12} \text{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T \\
 &\quad + \text{diag}[a(t), 0, 0]) \Gamma_\delta^\dagger O_{23}^T \\
 &= \frac{1}{2E} O_{23} \Gamma_\delta (\tilde{O}_{23} \tilde{O}_{13} \tilde{O}_{12} \text{diag}(0, \tilde{\Delta}_{21}, \tilde{\Delta}_{31}) \tilde{O}_{12}^T \tilde{O}_{13}^T \tilde{O}_{23}^T) \Gamma_\delta^\dagger O_{23}^T
 \end{aligned}$$

23 block of $O_{23} \Gamma_\delta \tilde{O}_{23}$

$$\begin{aligned}
 &= \begin{pmatrix} \cos \theta_{23} & \sin \theta_{23} \\ -\sin \theta_{23} & \cos \theta_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} \cos \tilde{\theta}_{23} & \sin \tilde{\theta}_{23} \\ -\sin \tilde{\theta}_{23} & \cos \tilde{\theta}_{23} \end{pmatrix} \\
 &= \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{i\psi} \end{pmatrix} \begin{pmatrix} \cos \tilde{\theta}_{23} & \sin \tilde{\theta}_{23} \\ -\sin \tilde{\theta}_{23} & \cos \tilde{\theta}_{23} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\tilde{\delta}} \end{pmatrix} \\
 &U_{\eta, \psi} = \text{diag}(1, e^{i\eta}, e^{i\psi})
 \end{aligned}$$

$$O_{23} \Gamma_\delta \tilde{O}_{23} = U_{\eta, \psi} \tilde{O}_{23} \Gamma_{\tilde{\delta}}$$

Finally

$$H = \frac{1}{2E} U_{\eta, \psi} \tilde{O}_{23} \Gamma_{\tilde{\delta}} \tilde{O}_{13} \tilde{O}_{12} \text{diag}(0, \tilde{\Delta}_{21}, \tilde{\Delta}_{31}) \tilde{O}_{12}^T \tilde{O}_{13}^T \Gamma_{\tilde{\delta}}^T \tilde{O}_{23}^T U_{\eta, \psi}^\dagger$$

Conclusions

- * Exact and Simple formula in constant matter

$$P_{ee} = C_{ee}$$

$$P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu}$$

$$P_{e\tau} = A_{e\tau}\cos\delta + B_{e\tau}\sin\delta + C_{e\tau}$$

$$P_{\mu\mu} = A_{\mu\mu}\cos\delta + C_{\mu\mu} + D_{\mu\mu}\cos 2\delta$$

$$P_{\tau\tau} = A_{\tau\tau}\cos\delta + C_{\tau\tau} + D_{\tau\tau}\cos 2\delta$$

$$P_{\mu\tau} = A_{\mu\tau}\cos\delta + B_{\mu\tau}\sin\delta + C_{\mu\tau} + D_{\mu\tau}\cos 2\delta$$

Exact formula for $A_{\alpha\beta}$, $B_{\alpha\beta}$ and $C_{\alpha\beta}$ has been derived

- * Overall feature of CP dependence in arbitrary matter

$$P_{ee} = C_{ee}$$

$$P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu}$$

$$P_{e\tau} = A_{e\tau}\cos\delta + B_{e\tau}\sin\delta + C_{e\tau}$$

$$P_{\mu\mu} = A_{\mu\mu}\cos\delta + B_{\mu\mu}\sin\delta + C_{\mu\mu} + D_{\mu\mu}\cos 2\delta + E_{\mu\mu}\sin 2\delta$$

$$P_{\tau\tau} = A_{\tau\tau}\cos\delta + B_{\tau\tau}\sin\delta + C_{\tau\tau} + D_{\tau\tau}\cos 2\delta + E_{\tau\tau}\sin 2\delta$$

$$P_{\mu\tau} = A_{\mu\tau}\cos\delta + B_{\mu\tau}\sin\delta + C_{\mu\tau} + D_{\mu\tau}\cos 2\delta + E_{\mu\tau}\sin 2\delta$$

- * CP dependence in symmetric matter reduces to same form as in constant matter