Validity of the Force-Field Equation to Describe Cosmic Ray Modulation

R.A. Caballero-Lopez[1], H. Moraal[2], and C.D. Steenberg[3]
(1) Institute for Physical Science and Technology, University of Maryland, USA
(2) School of Physics, Potchefstroom University, Potchefstroom, South Africa
(3) California Institute of Technology, Pasadena, California

Abstract

It is shown that the widely used Force-Field (FF) approximation to the cosmic ray transport equation (TPE) does not work well in the outer heliopshere, in particular for anomalous cosmic rays (ACRs). The even simpler Convection-Diffusion (CD) approximation produces better results in the outer heliosphere. Using a simple 1D solution to the TPE for comparison, we show that the widely used FF and CD approximations are poor in commonly occurring situations. The FF fares badly in the outer heliosphere and for ACRs in general, while the CD is only appropriate in the outer heliosphere. A full 1D solution is provided in the hope that it may be more commonly used instead of these approximations.

1. Introduction

The Force-Field approximation to the cosmic ray transport equation [1,2] is widely used, because it characterizes the entire modulation process by a single parameter, the so-called Force-Field (FF) potential, φ, ranging from about 300 to 1000 MV from solar minimum to solar maximum conditions. The cosmic ray transport equation for the evolution of the cosmic ray distribution function f in terms of particle momentum p, can be written in the two equivalent forms

$$\partial f/\partial t + \nabla \cdot (C\mathbf{V}f - \mathbf{K}\cdot\nabla f) + \frac{1}{3p^2}\frac{\partial}{\partial p}(p^3\mathbf{V}\cdot\nabla f) = Q \quad \text{and}$$

$$\partial f/\partial t + \nabla \cdot (\mathbf{V}f - \mathbf{K}\cdot\nabla f) - \frac{1}{3p^2}(\nabla\cdot\mathbf{V})\frac{\partial}{\partial p}(p^3 f) = Q$$

where $\mathbf{V}$ is the solar wind velocity and $\mathbf{K}$ is the diffusion tensor containing elements describing diffusion along the field, perpendicular to it, as well as gradient and curvature drifts. The quantity $C = -1/3(\partial \ln f/\partial \ln p)$ is the Compton-Getting factor. The FF approximation assumes that (a) there are no sources, $Q = 0$, (b) there is a steady state, $\partial f/\partial t = 0$, and (c) that the adiabatic energy loss rate $< dp/dt >= (p/3)\mathbf{V}\cdot\nabla f/f = 0$, so that the first form of the equation above reduces to $C\mathbf{V}f - \mathbf{K}\cdot\nabla f = \text{constant} = 0$. If there is also spherical symmetry, it

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reduces to $CVf - \kappa\partial f/\partial r = 0$, where the entire diffusion tensor $\mathbf{K}$ has collapsed to a single effective radial coefficient κ. When C is introduced explicitly, this equation becomes $(Vp/3)\partial f/\partial p + \kappa\partial f/\partial r = 0$, with solution $f(r,p) = f_b(r_b, p_b)$ along contours $dp/dr = Vp/3\kappa$. The subscript b designates values on the outer boundary of the modulation region. If the diffusion coefficient is separable in the form $\kappa = \beta\kappa_1(r)\kappa_2(p)$, the contours can be integrated to give

$$\int_p^{p_b} (\beta\kappa_2/p)dp = \int_r^{r_b} (V/3\kappa_1)dr \equiv \phi. \tag{1}$$

When $\kappa_2 \propto p$ and $\beta \approx 1$ the solution reduces to the very widely used form

$$p_b - p = \phi \approx 300 \text{ to } 1000 \text{ MeV/c}. \tag{2}$$

ϕ is a momentum loss, but it can alternatively be expressed as an energy loss or a modulation potential through the definition of rigidity by $P = pc/q = A/Z\sqrt{(T(T+2E_0)}$, where A and Z are mass and charge number, and T is kinetic energy per nucleon. We note that (a) it is ironic that one ends up with a modulation potential when the original assumption was that there are no energy losses, and (b) it is often forgotten that the simple FF potential in the form (2) only holds for the special rigidity dependence of κ. More generally, the FF parameter is actually κ_2/ϕ, as emphasized by [2].

An almost equivalent approximation follows from the second form of the transport equation if Q, $\partial f/\partial t$, and the third term (which is not the energy loss; see e.g. [4]) are again set equal to zero . This results in the so-called Convection-Diffusion (CD) equation, $Vf - \kappa\partial f/\partial r = 0$, with solution

$$f = f_b e^{-M}, \quad \text{where} \quad M = \int_r^{r_b} \frac{V dr}{\kappa} \tag{3}$$

The modulation function M is related to the FF parameter ϕ through $M = 3\phi/\beta\kappa_2$, but it is defined dimensionless.

2. Validity of the two approximations

The solid lines in Figure 1 show numerical solutions of the steady state spherically symmetric transport equation

$$3p^2\partial/\partial r[r^2(Vf - \kappa\partial f/\partial r)] - \partial/\partial r(r^2V)\partial/\partial p(p^3 f) = 0, \tag{4}$$

while the dashed and dotted lines are the equivalent FF and CD solutions (1) and (3). The intensities w.r.t. kinetic energy, $j = p^2 f$, and the radial intensity gradients, $g_r = (1/j)\partial j/\partial r$, are shown for both GCRs and ACRs. The outer boundary distance of the heliosphere was chosen at $r_b = 120$ AU, a radial diffusion mean free path $\lambda = 3\kappa/\nu = 0.25P$(GV) AU, with $\nu = \beta c$ the particle speed, and a solar wind speed $V = 400$ km/s were used. The combination of these parameters gives $M(1\text{ AU}) = 1.44/\beta P$, and $\phi(1\text{ AU}) = 474$ MV. The LIS for GCRs was chosen as $j_b = p^2 f_b = 10\beta/(T + E_0/2)^{2.6}$, while for ACR protons the form $j_b = (0.36/T)\exp[-0.189(T/0.36)^{2.029}]$ was used. The validity to represent ACR acceleration in the solar wind termination shock by such an effective spectrum on a passive outer boundary, was demonstrated by [3].

For GCRs in the inner heliosphere, the FF is quite good, as expected, because this is after all the situation for which it was developed. It is also a much better approximation than the CD, mainly because the latter makes no reference to energy losses. In the outer heliosphere however, *e.g.* 80 AU, the FF is not really superior to the CD. For ACRs, with their much steeper spectra on the other hand, the FF solutions in the inner heliosphere are entirely off scale, while the CD is far too steep and peaked. Both are very poor approximations to the numerical solution in this case. It is interesting, however, that in the outer heliosphere the CD approximation is much better than the FF approximation.

These effects are better demonstrated in the radial gradients of Figure 2 (left panel). For GCRs, the FF gradients, $g_r = CV/\kappa$, agree well with the true gradients at high energies, as is well-known, but the ACR FF gradients are about an order of magnitude too large at all energies. Notice that the gradient predicted by the CD is just the single dotted line given by $g_r = V/\kappa$. We note that the upturn in the ACR gradients at high energies is solely due to the spectral form there. In the right panel of Figure 2, the ACR spectrum on the boundary was modified so that it continues as a power law with spectral index -12.8 for $T >$ 190 MeV. Small changes on such steep spectra clearly have a huge effect on the gradients.

3. Numerical Solution

A short (12 lines long) Crank-Nicholson solution of the 1D transport equation (4) is made freely available for download at the address below. It is recommended for use where the FF is currently employed. It is as simple to use as the FF, and because it includes energy losses correctly, it gives a more reliable representation of the modulation.

http://www.puk.ac.za/physics/Physics%20Web/Research/mod1Dsimple.f

4. References

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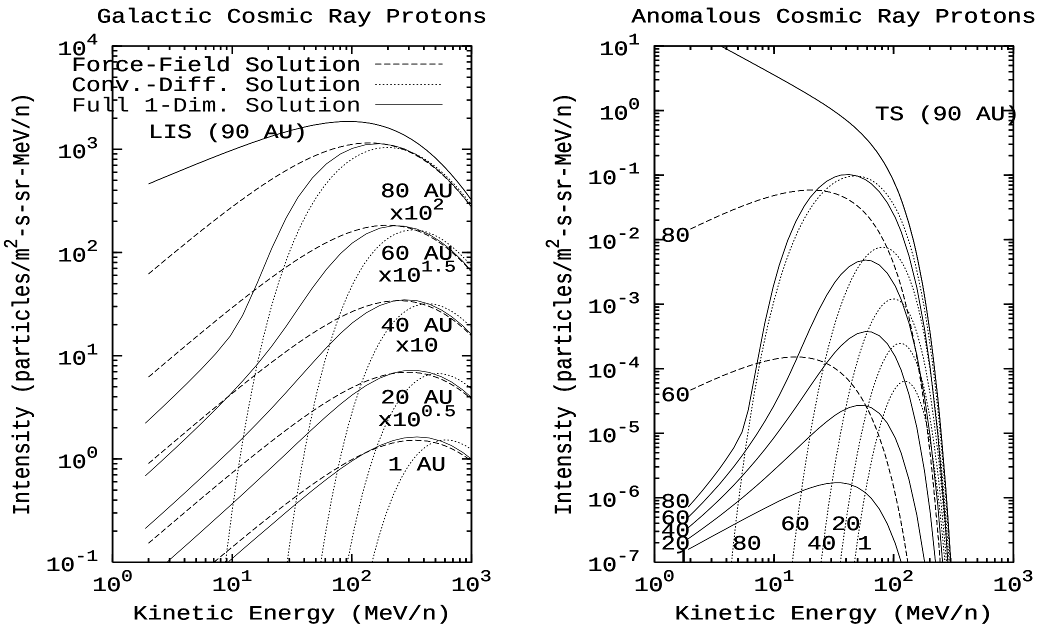

Fig. 1. Full 1-D, Force-Field and Convection-Diffusion solutions of the transport equation for GCR (multiplied by factors of $10^{0.5}$) and ACR protons in a heliosphere with r_b = 90 AU, V = 400 km/s, $\lambda = 0.25P$(GV) AU, $\kappa = \lambda v/3$, and ϕ(1 AU) = 474 MV.

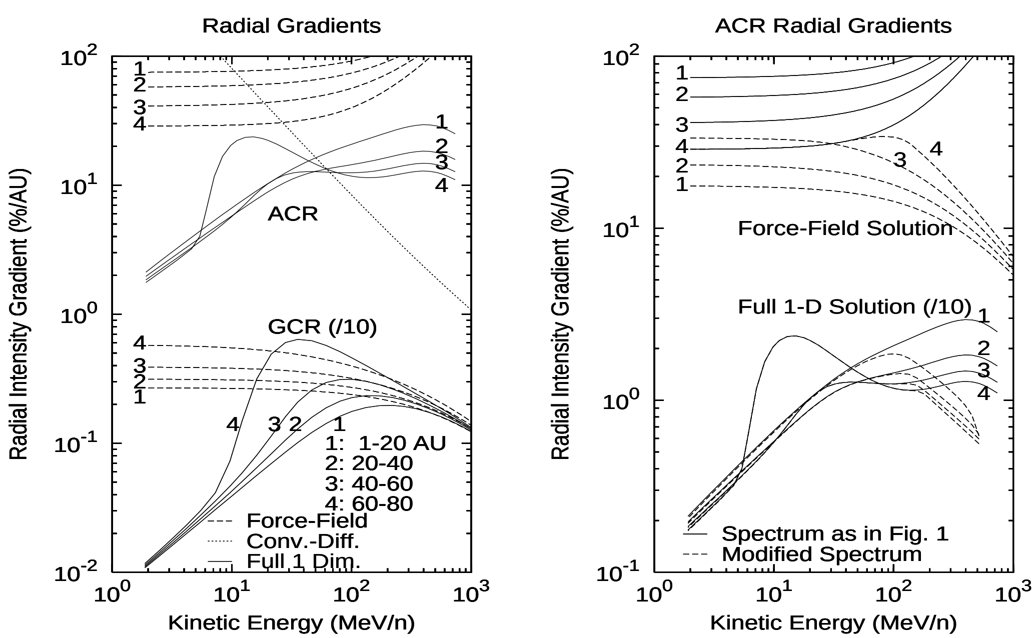

Fig. 2. Radial intensity gradients, calculated as $g = \ln(j_2/j_1)/(r_2 - r_1)$, for GCR and ACR protons (left), and ACR gradients for a varied boundary spectrum (right).