The second order pitch-angle approximation for the cosmic ray Fokker-Planck kinetic equations

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Abstract

The diffusive particle propagation and its pitch angle scattering is studied using kinetic equation of the Fokker - Planck form. The case is considered when charged particles preferable propagate along the strong mean magnetic field direction and undergo the pitch angle scattering with respect to it. The paper deals with solution of the equation for particle distribution function in the second order approximation in the pitch angle. The exact analytical solution is obtained in an integral form. The well known solution in the first order pitch angle approximation can be restored performing the small time limit in the result. Unlike the first order solution the obtained solution in the second approximation rightly shows that the pitch angle diffusion is closely connected with the particle transport along the mean magnetic field.

1. Introduction

Study of multiple charged particle scattering in magnetic field with random inhomogeneities as scattering centers is important in turbulent theory plasma [1], in problems of cosmic ray particle propagation through cosmic media [2, 3], and many other problems of particle transport [4]. If the magnetic field is sufficiently strong that the Larmor radius of particle $R_L \ll \lambda$ (λ - the particle mean free path with respect to its scattering in magnetic field inhomogeneities), the averaging over particle spiral motion around the magnetic field can be performed, and one can restrict himself to a simple rectilinear system.

The diffusive particle propagation and its angular scattering along the mean magnetic field is governed by kinetic equation of the Fokker - Planck form, and the particle distribution function, f, depends only on location, x, the pitch angle and time, t. Note that the Fokker - Planck scattering represents the scattering of particles in continuously fluctuating fields. Introducing the dimensionless variables, $y = x/\lambda$, $\tau = vt/\lambda$, (v is the particle velocity) and $\mu = \cos\theta$, the kinetic

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equation for $f(y,\tau,\mu)$ reduces to the well known form [5, 6]:

$$\partial_\tau f + \mu\, \partial_y f = \partial_\mu (1-\mu^2)\, \partial_\mu f + \frac{1}{\lambda}\delta(y)\, \delta(\tau)\, \delta(\mu-\mu_0)\,. \tag{1}$$

Note that cross-field transport (i.e. perpendicular diffusion and drift, energy change, or, adiabatic focusing) is not included into the model.

2. The first order approximation

Using the Fourier transform in the space variable, y, and the Laplace transform in the time, τ, the Eq. (1) gives the ordinary differential equation that does not lead to known special functions [7], therefore, some approximation of the Eq. (1) is necessary. The simplest approximation corresponds to very small pitch angle, θ, when one can put $\sin\theta \to \theta$ and $\cos\theta \to 1$. In this case the function $f_1(y,\tau,\theta)$ in the first order approximation is well known [3]:

$$f_1(y,\tau,\theta,\theta_0) = \frac{1}{2\lambda\tau}\ \exp\left[-\frac{\theta^2+\theta_0^2}{4\tau}\right]\ I_0\left(\frac{\theta\theta_0}{2\tau}\right)\ \delta(y-\tau)\,, \tag{2}$$

where $I_0(x)$ is the zeroth order Bessel function of imaginary argument.

3. The second order approximation

In the second order pitch angle approximation one must hold also term of $O(\theta^2)$. It means that $\sin\theta \to \theta$ and $\cos\theta \to (1-\theta^2/2)$ and Eq. (1) for f_2 in the second order approximation reads

$$\partial_\tau f_2 + \left(1-\frac{\theta^2}{2}\right)\partial_y f_2 = \frac{1}{\theta}\ \partial_\theta\ \theta\ \partial_\theta f_2 + \frac{1}{\lambda}\delta(y)\ \delta(\tau)\ \frac{\delta(\theta-\theta_0)}{\theta_0}\,. \tag{3}$$

The resulting solution takes the form [8]

$$\begin{aligned} f_2(y,\tau,\eta,\eta_0) \;=\; & \frac{1}{8\pi\lambda}\int_0^\infty \Big\{\exp\Big[-\mathrm{i}k(y-\tau) - (1+\mathrm{i})\sqrt{k}(\eta+\eta_0) \\ & \times\ \coth\Big((1+\mathrm{i})\sqrt{k}\tau\Big)\Big] \\ & \times\ I_0\left(\frac{(1+\mathrm{i})\sqrt{k}\sqrt{\eta\eta_0}}{2\sinh\big[(1+\mathrm{i})\sqrt{k}\tau\big]}\right)\frac{2(1+\mathrm{i})\sqrt{k}}{\sinh\big[(1+\mathrm{i})\sqrt{k}\tau\big]}\ +\ C.C.\Big\}\,dk\,, \end{aligned} \tag{4}$$

or in the variables $\{x,t,\theta,\theta_0\}$,

$$\begin{aligned} \widetilde{f_2}(x,t,\theta,\theta_0) \;=\; & \frac{1}{8\pi\lambda}\int_0^\infty \Big\{\exp\Big[-\frac{\mathrm{i}k}{\lambda}(x-vt) - (1+\mathrm{i})\frac{\sqrt{k}}{4}(\theta^2+\theta_0^2) \\ & \times\ \ \coth\Big((1+\mathrm{i})\sqrt{k}\frac{vt}{\lambda}\Big)\Big] \\ \times\ I_0\left(\frac{(1+\mathrm{i})\sqrt{k}\theta\theta_0}{2\sinh\big[(1+\mathrm{i})\sqrt{k}vt/\lambda\big]}\right) & \frac{2(1+\mathrm{i})\sqrt{k}}{\sinh\big[(1+\mathrm{i})\sqrt{k}vt/\lambda\big]} + C.C.\Big\}\,dk, \end{aligned} \tag{5}$$

where $C.C.$ denotes the complex conjugate term. This function is shown on Fig. 1.

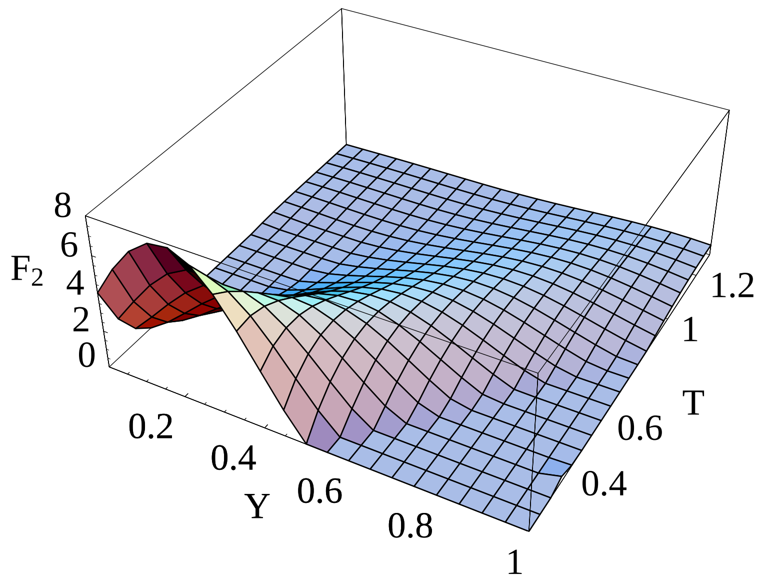

Fig. 1. *The space distribution $f_2(y,\tau)$ in the second approximation in the interval $\tau = 0.3 - 1.3$ for $\theta = 0.1$, $\theta_0 = 0$.*

Let one considers the expression (5) in non-zero but small time limit, $t \to 0$. Then the expression for $\widetilde{f_2}(x,t,\theta,\theta_0)$ acquires the form analogous to (2) obtained in the first order approximation,

$$\widetilde{f_2} \approx \frac{\lambda}{2vt} \, \exp\left[-\lambda\frac{(\theta^2+\theta_0^2)}{4vt}\right] \, I_0\left(\frac{\theta\theta_0\lambda}{2vt}\right) \, \delta\left(x - vt\left(1-\frac{\theta^2+\theta_0^2}{6}\right)\right). \tag{6}$$

One can see from the expression (6) that both processes of the pitch angle diffusion and the particle transport are here connected each other. Unlike the first approximation f_1 there is not "free" propagation in the small time limit solution as well as in the second approximation f_2 . So, the first approximation is suitable only in very small time interval, $\tau \ll 1$.

4. Discussion and conclusion

Unlike the first approximation, the function $f_2(\theta)$ describes the initially anisotropic stream during a larger time past the particle injection, and one has low level at $y \approx \tau$, especially for $y \gg 1$. The space distribution, $f_2(y)$, has been demonstrated on Fig. 1, where the pitch angle θ is fixed. The picture is similar for non-zero θ. The space distribution possess rather wide 'tail' behind the front of first particles at $y = \tau$. Its width increase with increasing time, and maximum decreases in amplitude and becomes later with increasing time. Temporal development of $f_2(y, \tau)$ is demonstrated on Fig. 2. We conclude that unlike the first

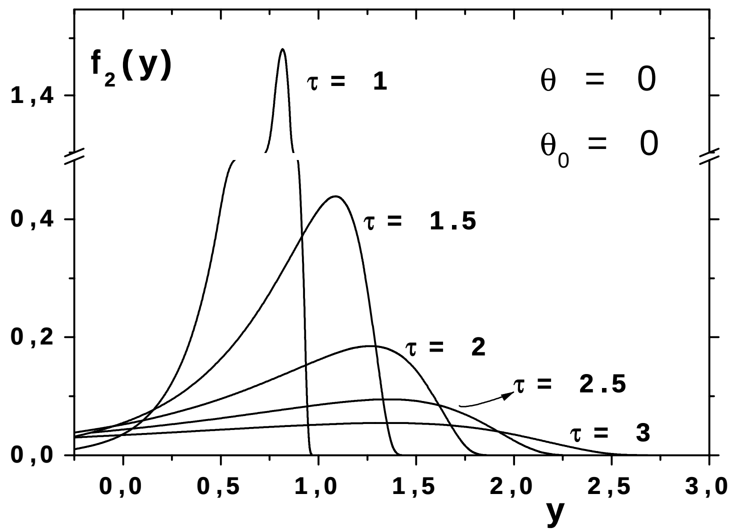

Fig. 2. *The space distribution $f_2(y)$ at time $\tau = 1, 1.5, 2, 2.5$ and 3 for $\theta = \theta_0 = 0$.*

approximation in pitch angle the derived expressions for the particle distribution function in the second approximation as well as the particle density gives more realistic picture of the pitch angle distribution after an unidirectional immediately particle injection.

Acknowledgements. The work was partially supported by INTAS, project No.00-810, and SAS, project No.7232.

5. References

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