Bondi Mass in Scalar Fields

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Abstract

The asymptotically flat space-time with scalar fields is studied. It shows that the concepts of Bondi mass, Bondi mass loss, etc., are also applicable to other fields, although they were originally defined by gravity. The generating formulae of Bondi mass loss and angular momentum loss by a dynamics Hamiltonian over a hyperboloid are given by the linearised theory.

1. Introduction

Applying the concept of quasi-local definition for gravity [1, 2, 3, 4], a dynamical Hamiltonian in an asymptotically flat space-time is discussed here. A hyperboloid is considered. Define Bondi mass by the Hamiltonian (not necessarily the gravity) on the hyperboloid and the Bondi mass loss can be interpreted if the Hamiltonian is not conserved in the time (or retarded) coordinate, i.e.,

$$\partial_0 \int \mathcal{H} dV \quad \neq \quad 0 \tag{1}$$

where $\mathcal{H}$ is the Hamiltonian density and dV is the integral region of the 3-dimensional volume over the hyperboloid. A characteristic Hamiltonian in the space-time such that, it is analysable, at least around the future null infinity, $\Im^+$, is derived, otherwise, a general form of the Hamiltonian is sought and its asymptotic behaviour is presented afterward. A simple case is considered here: a space-time with scalar fields. The corresponding Bondi energy is then studied by the linearised theory, and the generating formulae on a hyperboloid, including Bondi mass, Bondi mass loss, and angular momentum loss are then derived.

2. Minkowski Space-Time $\mathbb{M}$ with Scalar Fields

Consider Minkowski space-time with scalar fields φ. Under the coordinate transformation $(t, r, \theta, \phi) \mapsto (s, \rho, \theta, \phi) = (x^0, x^1, x^2, x^3)$ via

$$r = \rho^{-1} \quad , \quad t = s + \frac{\sqrt{1+\rho^2}}{\rho} \tag{2}$$

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the density of Lagrangian with φ on a hyperboloid Σ can be written as

$$\mathcal{L} = -\frac{1}{2}\rho^{-2}\sin\theta\left[-\frac{(\varphi_{,0})^2}{1+\rho^2} + \frac{2\varphi_{,0}\varphi_{,1}}{\sqrt{1+\rho^2}} + \rho^2(\varphi_{,1})^2 + \boldsymbol{r}^{AB}\varphi_{,A}\varphi_{,B}\right] \tag{3}$$

where the metric on a unit sphere $\boldsymbol{r}^{AB}$ satisfying $\boldsymbol{r}_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta\phi^2$ on $S(s,\rho) = \{x \in \Sigma_s | x^1 = \rho = constant\}$, in which, Σ_s is the cut of 4-manifold Σ by $s = constant$, i.e., $\Sigma_s = \{x \in \mathbb{M} | x^0 = s = constant\}$.

It is observed that the integral of such $\mathcal{L}$ is not convergent at infinity $\Im^+$. In order to remove such divergence, the renormalisation of $\mathcal{L}$ is considered by defining a new scalar function ψ such that $\psi = \rho^{-1}\varphi$. With this transformation, define a new Lagrangian $\tilde{\mathcal{L}}$ in a virtual space-time $\tilde{\mathbb{M}}$, such that

$$\tilde{\mathcal{L}} \;\; = \;\; \mathcal{L} + \frac{1}{2}\partial_0\left(\frac{\rho^{-1}\psi^2\sin\theta}{\sqrt{1+\rho^2}}\right) + \frac{1}{2}\partial_0(\sin\theta\psi^2) \tag{4}$$

It can be seen that this renormalised $\tilde{\mathcal{L}}$ also satisfies the variation principle in phase space ψ and has the same Euler-Lagrange equations of motion as $\mathcal{L}$ in Eq. (3) in phase space φ. $\tilde{\mathcal{L}}$ also gives the convergent integral at infinity. Two metrics with the transformation relation (2) have the relation $d\tilde{S}^2 = \rho^2 dS^2$.

Observing these two sets of space-time $(\tilde{\mathbb{M}}, \tilde{g})$ and $(\mathbb{M}, g)$ corresponding two Lagrangians $\tilde{\mathcal{L}}$ and $\mathcal{L}$ respectively, one can derive the following results.

Lemma 1 *ψ satisfies wave equation $\widetilde{\Box}\psi = 0$, where the conformal operator $\Box$ is defined as*

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\right) \tag{5}$$

Lemma 2 *If $\widetilde{\Box}\psi = 0$, then $\Box\varphi = 0$.*

Proofs for Lemma 1 and Lemma 2 are laborious but direct [11]. By these two lemmas, one knows that the conformal transformation is asymptotically invariant. Since the metric is asymptotically flat, the scalar curvature will therefore vanish.

3. Bondi Mass Loss at $\Im^+$

In order to analyse the energy change at $\Im^+$, one can therefore assume the Bondi mass is no longer conserved at $\Im^+$ so that a Dirichlet condition, $\varphi\mid_{\partial V} = f = f(t)$ on the boundary can hold. For the energy description, the energy on a hyperboloid Σ is defined. Although energy on a hyperboloid is not a general form of Hamiltonian, it plays an important role for the description of the radiation at $\Im^+$ [4], as this method is useful to the construction of the generation of the Poincaré group and could be applied to the angular momentum.

Now define the Hamiltonian density as $\tilde{\mathcal{H}} = \pi^0\psi_{,0} - \tilde{\mathcal{L}}$, in which, π^0 is time-momentum defined as $\pi^0 = \frac{\partial\tilde{\mathcal{L}}}{\partial\psi_{,0}}$. As shown above for $\tilde{\mathcal{L}}$, the Hamiltonian $\tilde{\mathcal{H}}$ is also analytic at $\Im^+$. The immediate question is: Is the Hamiltonian defined by the choice of coordinate transformation in (2) and the renormalised $\tilde{\mathcal{L}}$ in Eq. (4) relevant for the expression of energy? That is to say, one has to certify $\tilde{\mathcal{H}} = \mathcal{H}$, at least at $\Im^+$ for the analysis. Thus, another lemma should be derived:

Lemma 3 *Under the operation of the transformation of (2) and the renormalisation for $\tilde{\mathcal{L}}$ in (4), $\tilde{\mathcal{H}} = \mathcal{H}$ in the neighbourhood of $\Im^+$.*

Generally, $\tilde{\mathcal{H}}$ may differ from $\mathcal{H}$. However, by the conformal rescaling discussed above, $\tilde{\mathcal{H}} \simeq \mathcal{H}$ around $\Im^+$. Detailed proof can be found in Ref. [11].

With this scaled $\tilde{\mathcal{H}}$, the time-variation of $\tilde{\mathcal{H}}$ then reduces to

$$\partial_0 \int_\Sigma \tilde{\mathcal{H}} = \int_{\partial\Sigma} \pi^1\dot{\psi} \simeq \int_{S(s,0)} (-\sin\theta\dot{\psi}^2) \tag{6}$$

on the hyperboloid Σ and its boundary, $\partial\Sigma$, in which, $\partial\Sigma = S(s, \rho = 0)$ at infinity. Since the boundary term in Eq. (6) doesn't vanish, the Hamiltonian is not conserved in time either at $\Im^+$. Thus, Eq. (6) can be treated as the interpretation of Bondi mass loss.

4. Angular Momentum at $\Im^+$

When defining angular momentum, one can treat angular momentum along z-axis on Σ, J_z, as the generator for a Killing vector field $\frac{\partial}{\partial\phi}$, then shift it to $\Im^+$. By doing so, one can obtain

$$-\delta J_z = \int_\Sigma (\pi_{,\phi}\delta\psi - \psi_{,\phi}\delta\pi) = -\delta\int_\Sigma \pi\psi_{,\phi} + \int_\Sigma \pi\delta\psi_{,\phi} \tag{7}$$

Applying Euler-Lagrange equation of motion, the time-variation of the angular momentum, $\partial_0 J_z$, can be written as

$$-\partial_0 J_z = -\partial_0 \int_\Sigma \pi\psi_{,\phi} = \int_{\partial\Sigma} \pi^1\psi_{,\phi} \simeq \int_{S(s,0)} \sin\theta\dot{\psi}\psi_{,\phi} \tag{8}$$

Eq. (8) then interprets the angular momentum loss.

Observing Eqs. (6) and (8), one can formulate consequently

Theorem 1 *If Bondi mass is conserved at $\Im^+$, then the angular momentum J_z is also conserved.*

This result is an immediate consequence by setting $\partial_0 \int_\Sigma \tilde{\mathcal{H}} = 0$ in Eq. (6). This important consequence shows that, if without loss of Bondi mass, it is impossible to radiate away angular momentum [4], i.e., there is no news function [12], and the fields will be the same no matter to observe it by taking a spacelike hypersurface or by taking a null infinite hypersurface.

5. Conclusions

By the linearised theory, a dynamical Hamiltonian of scalar fields is analysed and reduced to gauge-invariant quantity in a quasi-local orientation. The quasi-local gauge-invariant Hamiltonian density is then derived on a hyperboloid and at the future null infinity $\Im^+$, giving the formulae of the Bondi mass loss and the angular momentum loss. Interests could then extend to other fields, e.g., scalar fields, matter fields, the Cosmological constant Λ, etc. If, such a definition still keeps its applicability, together with the quasi-local definition of gravitational energy [2], the total Hamiltonian can then be written. Since the Hamiltonians discussed by Hayward [2] and this work are physically well defined, it could be natural that the resultant total Hamiltonian should satisfy properties such as the positivity, or the monotonicity, or both, under some physical conditions, e.g., dominant energy condition, etc. However, this will require more detailed discussions.

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