Graviton Production by a Thermal Bath

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Abstract

Thermal fluctuations in the early universe plasma and in very hot astrophysical objects are an unavoidable source of gravitational waves (GW). Differently from previous studies on the subject, we approach this problem using methods based on field theory at finite temperature. Such an approach allows to probe the infrared region of the spectrum where dissipative effects are dominant. Incidentally, this region is the most interesting from the point of view of the detectability perspectives. We find strong deviations from a Planck spectrum.

1. Introduction

It is well known that thermal collisions in a plasma give rise to production of gravitational radiation [10]. Graviton emission is a consequence of particle acceleration in the scattering process (gravitational Bremsstrahlung). The emitted power of a non-relativistic gas was determined by Weinberg [9] by summing up the radiated energies per collision under the condition that the wave frequency is much larger than the mean collision frequency ($\omega \gg \omega_c$). This is required for the waves do not interfere. As was noted in [10], for $\omega \ll \omega_c$ the gas behaves as a fluid rather then as a collection of independent particles. The low frequency limit is, however, the most interesting from the point of view of GW detection perspective. In this contribution we address the problem of computing the gravitational emission rate by a relativistic plasma in the low frequency limit. We mainly focus on the contribution to the emission process of collective excitations of the plasma. For definiteness we consider here the case of a QED plasma.

2. Collective excitations in a relativistic plasma

Fluctuations of electric and magnetic fields in a heat bath are related to the dispersive properties of the medium by the *fluctuation-dissipation theorem* [6]. In a field theory language this theorem states that the photon spectral function $\mathcal{A}_{\mu\nu}(q)$ is related to the retarded and advanced Green's functions through

$$\mathcal{A}_{\mu\nu}(q) \equiv i(1 - e^{\beta q_0})^{-1} \left[\mathcal{D}_{\mu\nu}(q_0 + i\epsilon, \mathbf{q}) - \mathcal{D}^*_{\mu\nu}(q_0 + i\epsilon, \mathbf{q})\right] \ , \tag{1}$$

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where q is the photon 4-momentum and $\beta = 1/T$ is the inverse temperature. In the Feynman gauge this quantity can be decomposed into transversal and longitudinal components $\mathcal{A}_{\mu\nu} = -P_{\mu\nu}\mathcal{A}_T - Q_{\mu\nu}\mathcal{A}_L$ where

$$\mathcal{A}_{T,L}(q) = -\frac{1}{\pi}\frac{\mathrm{Im}\,\Pi_{T,L}}{|q^2 - \Re\Pi_{T,L}|^2 + |\mathrm{Im}\,\Pi_{T,L}|^2} \quad . \tag{2}$$

The expressions of the transversal (longitudinal) components, $\Pi_{T(L)}$, of the polarization tensor, as well as the definitions of the projectors $P_{\mu\nu}$ and $Q_{\mu\nu}$, are given in [1]). The dispersion relation of transverse (longitudinal) modes is given by $q_o^2 - q^2 = \Re\Pi_{T(L)}(q)$. At small momenta $\Pi_{T(L)}(q)$ is of the order of the plasma frequency which for a relativistic plasma is $\omega_p \simeq eT/3$. The propagation of photons is strongly altered for $q_0,\ \mathbf{q} \lesssim \omega_p$; in this case they behave as quasi-particles endowed with an effective mass of order eT. Whereas plasma eigenmodes are distributed with a Planckian spectrum peaked at $q_0 \sim T$ fluctuations below the light cone ($q_0 < |\mathbf{q}|$) are subject to Landau damping which strongly affect their statistical properties. Using Eqs.(1,2) one finds [6]

$$\begin{aligned} \langle \mathbf{B}^2 \rangle_q &= \frac{2\pi}{1 - e^{-\beta q_0}} 2q^2 \mathcal{A}_T(q) \quad , \\ \langle \mathbf{E}^2 \rangle_q &= \frac{2\pi}{1 - e^{-\beta q_0}} \left[2q_0^2 \mathcal{A}_T(q) + q^2 \mathcal{A}_L(q) \right] \quad . \end{aligned} \tag{3}$$

In the following we will focus on magnetic fluctuations since electric fields are screened in the static limit. Interestingly, it was showed by Lemoine [6] that a large peak is present in the magnetic fluctuations spectrum at nearly zero-frequency. This can be interpreted as "squeezing" of energy toward $q_0 = 0$ because of the non-linear coupling at small momenta. The effect give rise to a random quasi-static magnetic field with root-mean-square energy at the scale l given by

$$B_l \simeq B_0 \left(\frac{l_p}{l} \right)^{3/2} \quad , \quad B_0 = \left(\frac{T\omega_p^3}{32\pi^{9/2}} \right)^{1/2} \quad , \tag{4}$$

where $l_P \equiv 2\pi/\omega_p$.

3. Gravitational waves production

It is well known that stochastic magnetic fields can power gravitational waves production [4]. Magnetic anisotropic stresses

$$\tau_{ij}^{(B)}(\mathbf{k}) = \frac{1}{4\pi} \int d^3q \, \left(B_i(\mathbf{q}) B_j^*(\mathbf{k}-\mathbf{q}) - \frac{1}{2} B_l(\mathbf{q}) B^{l\,*}(\mathbf{k}-\mathbf{q}) \delta_{ij} \right) \tag{5}$$

act as a source term in the time evolution equation for the spatial components of the metric perturbations. In the Fourier space this equation can be written

$$\ddot{h}_{ij} + k^2 h_{ij} = 8\pi G \Pi_{ij}^{(B)} \ , \tag{6}$$

where $\Pi_{ij}^{(B)} = \left(P_i^a P_j^b - \frac{1}{2}P_{ij}P_{ab}\right) \tau_{ab}^{(B)}$ is the tensor component of $\tau_{ij}^{(B)}$. Due to the stochastic nature of thermal fluctuations, hence also of the GW background they give rise to, the relevant quantity we are concerned here is the GW spectral function $\langle h^{ij}(\mathbf{k},t), h_{ij}(\mathbf{k}',t)\rangle$. Starting from the general solution of (6) $h_{ij}(\mathbf{k},t) = \dfrac{8\pi G}{k}\int_o^t dt'\Theta(t-t')\sin[k(t-t')]\Pi_{ij}(\mathbf{k}',t')$ and replacing ensemble average with time average, one finds [5,3]

$$\langle\, h_{ij}(\mathbf{k},t)\, h^*_{ij}(\mathbf{k}',t)\,\rangle \simeq \frac{9\sqrt{2}\,(16\pi G)^2\,\Delta t_*}{16\,k^3}\delta^3(\mathbf{k}-\mathbf{k}')\int d^3q\, B^2(q)B^2(|\mathbf{k}-\mathbf{q}|) \quad (7)$$

where Δt is the source duration. Assuming a power-law spectrum $B^2(k) = Ak^n$ for the fluctuations, it can be showed [5,2] that whenever $n > -3/2$ the convolution is dominated by the frequency independent term $\int d^3q\, B^2(q)B^2(|\mathbf{k}-\mathbf{q}|) \simeq 4\pi A^2\dfrac{k_{max}^{2n+3}}{2n+3}$. This give rise to a white-noise GW spectrum. In the previous section we showed that the energy density in quasi-static magnetic fluctuations scales with the scale l as $B_l^2 \propto l^{-3/2}$. Since $B_l^2 \sim B^2(k)k^3|_{k=1/l} \propto l^{-(n+3)}$ it follows that $n=0$ in this case, which fulfills the $n > -3/2$ condition. It was recently claimed by the authors of [2] that $n=0$ is unphysical and it should be replaced by $n=2$. Such substitution would not affect our results. In our case k_{max} has to be identified with the plasma frequency ω_p and k_{min} (see below) with the Hubble expansion rate $H(T)$.

The real-space correlator is given by

$$\langle\, h_{ij}(\mathbf{x},t)\, h^{ij}(\mathbf{x},t)\,\rangle = \frac{V^2}{(2\pi)^6}\int d^3k d^3k' e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}\langle\, h_{ij}(\mathbf{k},t)\, h^{ij\,*}(\mathbf{k}',t)\,\rangle\,. \quad (8)$$

The power-spectrum normalization requires $A = \dfrac{B_0^2}{8\pi}\dfrac{\pi^2}{V}(n+3)\,\omega_p^{-(n+3)}$. Then, for $n=0$ we find

$$\langle\, h_{ij}(\mathbf{x},t)\, h_{ij}(\mathbf{x},t)\,\rangle = \frac{81\sqrt{2}}{32}\, G^2\,\Delta t B_0^4\omega_P^{-3}\delta^3(\mathbf{k}-\mathbf{k}')\int_H^{\omega_P/2\pi}\frac{dk}{k}\,. \quad (9)$$

The GW power spectrum is usually parametrized in terms of the *characteristic amplitude* $h_c(f)$ of GWs at frequency f, defined by [8] $\langle\, h_{ij}(\mathbf{x},t)\, h_{ij}(\mathbf{x},t)\,\rangle \equiv 2\displaystyle\int_0^\infty\frac{df}{f}h_c^2(f,t)$. The characteristic amplitude measured today at a frequency f is

$$h_c(k_0,t_0) = \frac{a}{a_0}h_c(k=\frac{a_0}{a}k_0,t) \simeq 10^{-16}\left(\frac{100}{g_*}\right)^{1/3}\left(\frac{\mathrm{TeV}}{T}\right)\,. \quad (10)$$

It is natural to assume the emission time interval to be $\Delta t \sim H^{-1} = g_*^{-1/2}\frac{M_P}{T^2}$, where M_P is the Planck mass and T the emission temperature. Disregarding the weak dependence on the number of relativistic d.o.f. g_* we finally find

$$h_c(f, t_0) \simeq 10^{-24} \kappa^3 \left(\frac{T}{M_P} \right)^3 , \qquad (11)$$

where we parametrized the plasma frequency dependence on T as $\omega_P = \kappa T$. For QED $\kappa \simeq 0.1$ but it can be larger for GUT theories. Furthermore the larger number of gauge bosons predicted by these theories may very well give rise to more than a factor 10 enhancement in the predicted value of h_c. The minimal wavenumber today corresponds to the red-shifted value of $H(T)$. In term of frequency it is $f_{min} \simeq 10^{-4}\ \mathrm{Hz} \left(\frac{T}{\mathrm{TeV}} \right)$. It is remarkable that the expected GW signal may be in the detectable range of LISA [7] if gravity becomes strong ($M_P \sim TeV$) at the TeV scale.

Independently on these speculations, our main result is to have shown that collective excitations in a relativistic plasma give rise to a GW background with a frequency independent amplitude. In the range of detectable frequencies this background overwhelm that produced by the conventional gravitational bremsstrahlung.

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