Variational Principle For Fokker-Planck Cosmic Rays Transport Equation

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Abstract

Classical field theory is the generalization of particle mechanics and continuous systems. Typical situations involve a continuous approach suitable for fluid mechanics, electromagnetism and general relativity. Although the fields will provide all the physical information that's needed for observation , a theory with the intent to describe the nature must be available to explain the relation between field quantities and observables like energy, momentum and the flows therefore. This work is the propose a Lagrangian density for obtaining the Fokker-Planck cosmic rays transport equation. By creating this "cosmic ray transport field", it is possible to apply the Noether's theorem to find the energy-momentum tensor and currents of a single cosmic ray source.

1. Introduction

The spectrum of cosmic rays can be approximately described by a single power law with index -3 from $\sim 10\,GeV$ to the highes energies ever observed $\sim 10^{20}\ eV$. The only feature observed below $10^{18}\ eV$ is a knee around $10^{15}\ eV$. Because of this featureless spectrum, it is believed that cosmic ray production and propagation is governed by the same mechanism over decades of energy, the same mechanism at least works below the knee and the same or another one works above the knee. Meanwhile the origin of the cosmic rays spectrum is not still understood [1].

The sources of cosmic rays are believed to be novae and supernovae remnants, pulsars, compact objects in close binary systems and stellar winds. Observations of X-ray and γ-ray emission from these objects reveal the presence of energetic particles thus testifying to efficient acceleration processes near these objects.

To extract information which is contained in cosmic ray abundances and γ-ray fluxes one need to develope a model of particle production and propagation in the galaxy. Analytical and semi-analytical models are able to interpret one or only a few features and often fail when they try to deal with the whole variety of data. The goal of this work is present the characteristics of Fokker-Planck cosmic

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ray transport equation using a variational method, that means a Lagrangian function, Euler-Lagrange motion equations and the Noether's theorem, with this tools is possible to find the conservative quantities like energy-momentum tensor and currents of this field.

The Noether's theorem for conservation currents is expresed by [2] :

$$J^{\nu} = -i(\frac{\partial L}{\partial u_{,\nu}} u - \frac{\partial L}{\partial u^{+}_{,\nu}} u^{+}) \tag{1}$$

so:

$$\frac{\partial J^{\rho}}{\partial x^{\rho}} = 0 \tag{2}$$

and also the momentum-energy tensor:

$$T^{\nu}_{\alpha} = -L\delta^{\nu}_{\alpha} + \frac{\partial L}{\partial u^{A}_{,\nu}} u^{A}{}_{,\alpha} \tag{3}$$

2. The Fokker-Planck Cosmic Ray Transport Field

Is possible define a field given by $\psi = \psi(x^{\mu})$ and the complex conjugate $\psi^{+}(x^{\mu})$ with parameters $x^{\mu} = x^0, x^1, x^2, x^3, x^4$, where $x^0 = ct$, $x^{1,2,3} = x, y, z$ and $x^4 = bp$ where p is the momentum modulus, b a dimensional constant and x, y, z the position coordinates, so we define the latin index i, j from 1 to 3 and the greek index μ, ν, ρ from 0 to 4 in analogy of the relativity theory. In this model the diffusion tensor $\kappa^{\nu}_{\mu}(x^i, x^4)$ is defined by :

$$\kappa^{\nu}_{\mu} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \kappa^{i}_{j} & 0 \\ 0 & 0 & \kappa^{4}_{4} \end{bmatrix} \tag{4}$$

where $\kappa^{i}_{j} = \kappa^{i}_{j}(x, y, z)$ the spatial diffusion coeffcient, $\kappa^{4}_{4} = \alpha K(p)$ with α a dimensional constant and $K(p)$ is the momentum-space diffusion coeffcient and the product $\psi(x^{\mu})^{+}\ \psi(x^{\mu})$ is the density per unit of total particle momentum.

The lagrangian of Fokker-Planck equation proposed is :

$$L_{FP} = L_{Diff} + L_{Source} + L_{Conv} \tag{5}$$

where g is a dimensional constant and i the imaginary unit :

$$L_{Diff} = \frac{i}{2} g\kappa^{\nu}_{\mu}(\frac{\psi}{\psi^{+}}\partial^{\mu}\psi^{+}\partial_{\nu}\psi^{+} - \frac{\psi^{+}}{\psi}\partial^{\mu}\psi\partial_{\nu}\psi) \tag{6}$$

$$L_{Source} = \frac{i}{2} gQ_{\mu}(\frac{1}{\psi}\partial^{\mu}\psi - \frac{1}{\psi^{+}}\partial^{\mu}\psi^{+}) \tag{7}$$

$$L_{Conv} = \frac{i}{2} g a_\mu (\psi^+ \partial^\mu \psi - \psi \partial^\mu \psi^+) \tag{8}$$

The vector a_μ is defined as:

$$a_\mu = [c, v_i, N x^4 + \frac{2}{x^4}\kappa_4^4 - \frac{x^4}{3}\frac{\partial v_i}{\partial x^i}] \tag{9}$$

where v is the convection velocity and the source term :

$$Q_\mu = [Q(t), Q(x, y, z), Q(p)] \tag{10}$$

Now, using the Euler-Lagrange equations we obtain the motion equations :

$$\begin{aligned} &\partial_\rho(a^\rho\psi - 2\tfrac{\psi}{\psi^+}\kappa^\rho_\mu \partial^\mu \psi^+) + a_\mu \partial^\mu \psi^+ - \kappa^\rho_\mu \psi (\tfrac{\partial^\mu \psi^+}{\psi^+}\tfrac{\partial_\rho \psi^+}{\psi^+} + \tfrac{\partial^\mu \psi}{\psi}\tfrac{\partial_\rho \psi}{\psi}) = -\psi \partial_\rho Q^\rho \\ &\partial_\rho(a^\rho\psi^+ - 2\tfrac{\psi^+}{\psi}\kappa^\rho_\mu \partial^\mu \psi) + a_\mu \partial^\mu \psi - \kappa^\rho_\mu \psi^+ (\tfrac{\partial^\mu \psi}{\psi}\tfrac{\partial_\rho \psi}{\psi} + \tfrac{\partial^\mu \psi^+}{\psi^+}\tfrac{\partial_\rho \psi^+}{\psi^+}) = -\psi^+ \partial_\rho Q^\rho \end{aligned} \tag{11}$$

Now, using (5) into the expresion of Noether's theorem for complex fields given by (1) we obtain:

$$\frac{\partial J^\rho}{\partial x^\rho} = \frac{\partial}{\partial x^\rho}(a^\rho W - \kappa^\rho_\mu \partial^\mu W - Q^\mu) \tag{12}$$

where $W = \psi^+\psi$.

Now, using the vector (9) in (12) we find the Fokker-Planck equation:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x^i}(v_i W - \kappa^i_j \frac{\partial W}{\partial x^j}) - \frac{\partial}{\partial p}(p^2 D_{pp}\frac{\partial}{\partial p}(\frac{W}{p^2}) - \frac{p}{3}W\frac{\partial v_i}{\partial x^i}) + NW = q(r, p, t) \tag{13}$$

where $D_{pp} = \kappa_4^4$ and $N = (\frac{1}{\tau_f} - \frac{1}{\tau_r})$, $q(r, p, t) = \frac{\partial Q}{c\partial t} + \frac{\partial Q}{\partial r} + \frac{\partial Q}{b\partial p}$, with τ_f the time scale for fragmentation and τ_r the time scale for the radioactivity decay .

Now, in the equation (12), for index $\rho = 4$:

$$\frac{\partial}{\partial p}[(Nbp + \frac{2\alpha K(p)}{bp} - \frac{bp}{3}\frac{\partial v_i}{\partial x^i})(\psi^+\psi) - \frac{\alpha K(p)}{bp}\frac{\partial(\psi^+\psi)}{\partial p} - Q(p)] = 0 \tag{14}$$

If we consider the convection velocity v as a constant and $N \to 0$, the equation (14) have the solution:

$$\psi^+\psi = -bp^2 \int \frac{Q(p)}{\alpha K(p)} p^{-2} dp \tag{15}$$

An inmediatly result is the energy current given by eq.(15), making the assumption $K(p) = Ko$ constant for $\sim 1\ Kpc$ [3] and taking in account that experimentally the source has a exponential dependence of the form:

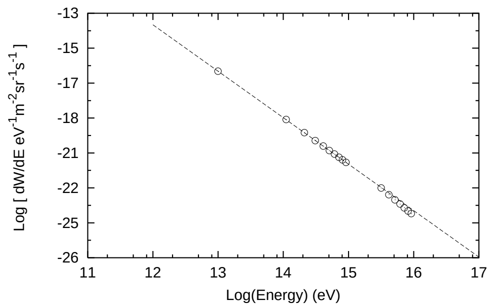

Fig. 1. Cosmic rays spectrum fitted for equation (17) with values γ=3.66 and κ_o=37.23.

$$Q(p) = \int \varepsilon p^{-\gamma} dp \tag{16}$$

where ε a dimensional constant and (12), we obtain :

$$\frac{\partial(\psi^{+}\psi)}{\partial E} = \frac{(\gamma-2)}{\gamma(\gamma-1)Ko}\left(\frac{E}{c}\right)^{1-\gamma} \tag{17}$$

The figure (1) shows the cosmic ray spectrum and plot of equation (17).

3. Conclusions

The Fokker-Planck transport equation for cosmic rays have a variational principle given in this model by lagrangian function (5), this lagrangian is invariant under $U(1)$, the interpretation for the elements of momentum-energy tensor of this field and the motion equations (11) is not well understood. In the calculations, the index covariant and contravariant are the same because is assumed a orthogonal space, then the difussion tensor (4) shows that is possible the inclusion of spatial diffusion and momentum-space diffusion in only one expresion.

Acknowledgements. This work is supported by the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

1. Strong A.W., Moskalenko I.V,Astrophys.J.,509,212-228,1998.
2. Sokolov A.A.,Ternov I.M., Zhukovski V.CH, Borisov A.V., Quantum Electrodynamics, MIR-Moscu, 1989.
3. Webb G.M., Gleeson L.J.,Propagation of Cosmic Rays, 1978.