Hênon-Heiles type Hamiltonian in Cosmological Perspective

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Abstract

Cosmic rays may provide some physical realization of the Henon-Heiles type Hamiltonian model which is otherwise wild enough. Keeping this view in mind, we investigate the Henon-Heiles type Hamiltonian with indefinite kinetic energy term. Though it is oversimplified, imposing cosmological constraint and taking the advantage of indefinite kinetic energy term the Hamiltonian can be proposed for general relativistic situation. Thus we are able to find out some new integrability conditions of the Hamiltonian.

1. Introduction

The concept of Cosmological constraint, that is, zero total energy (Kinetic + Potential) is implicit with the popular model for the origin of the universe. In this model universe forms out of a metastable false vacuum which undergoes inflation. In principle, this formation may be possible by the head-on interaction of two ultra energetic cosmic ray particles. Perhaps two cosmic ray protons each with the Plank energy (10^{27} eV) will be sufficient [5]. Being motivated by this idea, the concept of Cosmological constraint can be applied in the discussion of Hênon-Heiles type Hamiltonian which is extensively studied in nonlinear dynamics but still lacking of proper physical realization.

2. Hamiltonian with Indefinite Kinetic Energy Form

Hênon-Heiles model Hamiltonian was first introduced as a model for the motion of a star inside a galaxy[1]. This Hamiltonian can also be interpreted as a model for a single particle moving in two dimension under the action of a force described by a potential energy function. The generalized form of the Hênon-Heiles type Hamiltonian can be written as[2]

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(Ax^2 + By^2) + (\frac{1}{3}Cy^3 + Dx^2y) \tag{1}$$

where A, B, C and D are (real) parameters, x and y are the spatial coordinates, and p_x and p_y are the corresponding conjugate moementa. But in our proposed

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situation, at least for the toy-representation, in addition to the nonlinear character the Hamiltonian should be in pseudo-Riemannian space. This condition can be satisfied if we find the Hamiltonian with indefinite kinetic energy form and impose cosmological constraint. Eventually Hamiltonian structurally similar to(1) but with indefinite kinetic energy form is found in different contexts[2]. For our purpose we consider a general Hamiltonian (with arbitrary parameters A, B, C and D) of the form which can also represent coupled upper-hybrid and magnetoacoustic waves including both positive and negative group dispersion:

$$H = \frac{1}{2}({\pi_E}^2 + p{\pi_N}^2) + \frac{1}{2}(AE^2 + BN^2) + \frac{1}{3}(CN^3 + DNE^2) \tag{2}$$

where $p = \pm$ and $\pi_E \equiv \frac{dE}{d\xi}$, $\pi_N \equiv \frac{dN}{d\xi}$ are, respectively, the cannonical momenta conjugate to E and N. When p=-1, that is, for positive group dispersion the kinetic energy term is indefinite.
(ii) When the dependent variable of the usual KDV equation is made complex, the resulting form of the equation can be dervied from a Hamiltonian having indefinite kinetic energy term.
(iii) Similarly, if the dependent variable is made complex, the Hamiltonian representing classical dynamical systems with one-degree of freedom becomes

$$H(q,p) \rightarrow H_1 + iH_2 \equiv \left[\frac{1}{2}(p_1^2 - p_2^2) + V_1(q_1,q_2)\right] + i\left[p_1 p_2 + V_2(q_1,q_2)\right]$$

The kinetic energy term in H_1 is not positive definite.

3. Hamiltonian in Pseudo-Riemannian Space

Maupertius principle allows one to reduce the Hamiltonian flows to the geodesic flows on Riemannian spaces equipped with the Jacobi metrices. Then the problem of local instability (integrability) appears as the problem of studying the geodesic deviation equation[3]. Therefore , we make an attempt to consider the Hamiltonian in Riemannian geometric context. As the kinetic energy form of the Hamiltonian(2), with p=-1, is indefinite, the case can be treated as pseudo-Riemannian [3]. This inspires one to study instability(integrability)of a dynamical system through geometrical method. The Hamiltonian attributing the Riemannian space can be taken in the form

$$H(p,q) = \frac{1}{2}\eta_{\alpha\beta}p^\alpha p^\beta + V(q)$$

Thereafter, through the Jacobi metric

$$g_{\alpha\beta} = 2|(h - V)|\eta_{\alpha\beta}$$

with parameter s(t) along a geodesic where $\frac{ds}{dt} = 2|h - V|$, and h , the total energy function , the Hamiltonian flows can be reduced to the geodesic flow in Riemannian space. Assuming Hamiltonian constraint ie h=0 to characterise pseudo-Riemannian space we can have

$$g_{\alpha\beta} = 2|V|\eta_{\alpha\beta} \tag{3}$$

Here ,

$$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

and,

$$V = \frac{1}{2}\left(AE^2 + BN^2\right) + \frac{1}{3}\left(CN^3 + DNE^2\right) \tag{5}$$

4. Integrability through Gaussian Curvature

Geodesic deviation equation corresponding to a dynamical system in a general two-dimensional case can be expressed as [3]

$$\frac{D^2 n}{ds^2} + \hat{K}n = 0 \tag{6}$$

where n is geodesic deviation vector and $\hat{K}$ is the Gaussian curvature. Now it is easily observed that the geodesic flow is locally unstable if the Ricci scalar (Gaussian curvature, as$\hat{K} = \frac{1}{2}\hat{R}$, where $\hat{R}$ is the Ricci scalar)is negative[4]. Moreover , $\hat{R} = 0$, reflects the integrability of the model [4].
We write the Hamiltonian(2) in terms of $p(p_1, p_2)$ and $q(q_1, q_2)$ as

$$H = \frac{1}{2}\left(p_1^2 + pp_2^2\right) + \frac{1}{2}\left(Aq_1^2 + Bq_2\right) + \frac{1}{3}\left(Cq_2^3 + Dq_1^2 q_2\right) \tag{7}$$

The Jacobi metric is

$$g_{\alpha\beta} = 2|V(q_1, q_2)|\eta_{\alpha\beta}$$

where $\eta_{\alpha\beta} = dig\,(1, -1)$

$$\Longrightarrow g^{\alpha\beta} = \frac{1}{2|V(q_1, q_2)|}\eta^{\alpha\beta} \tag{8}$$

Here,

$$V = \frac{1}{2}\left(Aq_1^2 + Bq_2^2\right) + \left(\frac{1}{3}Cq_2^3 + Dq_1^2 q_2\right) \tag{9}$$

$$g_{\alpha\beta} = 2\left|\frac{1}{2}\left(Aq_1^2 + Bq_2^2\right) + \left(\frac{1}{3}Cq_2^3 + Dq_1^2 q_2\right)\right|\eta_{\alpha\beta}$$

Gaussian curvature $\hat{K}$ is given by

$$\hat{K} = \frac{R_{hijk}}{g_{hj}g_{ik} - g_{kk}g_{ij}}$$

Here the only nonvanishing $R_{hijk}$is R_{1212} due to(4).

$$R_{1212}=\frac{\begin{array}{l}-\frac{1}{2}A\left(A+b\right)q_1^2+\frac{1}{2}B\left(A+B\right)q_2^2\\+\left(\frac{1}{3}AC-\frac{2}{3}C^2+\frac{2}{3}BC+\frac{2}{3}CD+BD\right)q_2^3\\+\left(\frac{5}{3}BD-\frac{8}{3}AD-AC\right)q_1^2q_2+\left(-\frac{10}{3}D^2+\frac{4}{3}CD\right)q_1^2q_2^2+D^2q_1^4+C^2q_2^4\end{array}}{V} \tag{10}$$

For complete integrability of the Hamiltonian(2), the folloing cases are reported by [2] : Case I: Arbitrary A, B and C=pD Case II: B=pA and C=pD Case III: B=16pA and C=16pD. In our approach , we investigate integrable cases subject to $R_{1212}=0$ with $p=-1$. We are not able to recover the case (I) and case (III) even in particular context. But for case(II) , that is, when $B=-A, C=-D$, the Hamiltonian is integrable provided $q_2^2=-5q_1^2$ and, $q_1=-\frac{5i\sqrt{5}}{24}$ The result seems interesting because (q_1+q_2) gives a complex number and complexification of the dependent variable offers a Hamiltonian with indefinite kinetic energy term, representing classical dynamical system with one degree of freedom[2].

As the reported cases of integrability [2] shows $C\propto pD$, hence for p=-1, C and D should have opposite signs. But for the coupled waves that can be represented by the Hamiltonian(2), the parameters C and D are, by definition,positive definite. Hence, it is reported by Rao[2] that if the above result strictly holds good, then, it appears that there are no integrable cases for the coupled upper-hybrid and magnetoacoustic waves when the group dispersion is positive. However, in our approach, for A=-B, C=D(positive), Hamiltonian becomes integrable provided $q_2=2\sqrt{2}q_1$ and $q_1=-\frac{75}{26}$. Interestingly, in this case, (q_1+q_2) is not a complex number.

5. Conclusion

In the context of the motivating idea, our works can only be viewed as toy model. But Einstein's astonishing principle, "field strength=curvature" has been extended by physicists, and now all the field strengths occurring in elementary particle physics(which are required in order to construct a Lagrangian) are discussed in terms of curvature and connections but it is the curvature of a vector bundle, that is, the field space that arises not the curvature of space time. So if the assumed interaction is between two protons, and integrability is searched through curvature, perhaps the consideration of the idea of vector bundle will be proper.

6. References

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