Mechanism of Molière Expansion for the Angular Distribution and Improved Molière Functions Evaluated from the Single-Scattering Splitting Model

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Abstract

We have attempted to derive the Molière angular distribution of multiple Coulomb scattering through the splitting cross-section method. The separation of cross-section at $\chi_{\rm B}$ or $e^{B/2}$ times the screening angle well reconstructs the Molière expansion as the first approximation, although a little distortion from the gaussian is seen in the central distribution due to the higher approximation terms. We have found well-known characteristic angle $\chi_{\rm C}$ plays good discriminating angle to give central gaussian distribution and low-frequent large-angle scattering. We show the angular distribution derived through this splitting method shows good agreement with that derived by Molière-Bethe theory.

1. Introduction

The Molière expansion of angular distribution was well explained by the divided cross-section model [1], separation of the single scattering into the moderate scattering and the large-angle scattering at an adequate separation angle. The high-frequent moderate scattering gives central normal distribution, and null and additional low-frequent large-angle scatterings give the initiating gaussian term and subsequent Molière terms. The splitting of cross-section at $\chi_{\rm B}$ or $e^{B/2}$ times the screening angle is adequate in the first order of approximation. We examined whether the Molière expansion is reconstructed by the splitting, and found the penetrating depth is not thick enough for the moderate scattering to form the gaussian distribution. We searched for some smaller discriminating angle and have found $\chi_{\rm C}$ is a good discriminating angle for the moderate scattering to produce gaussian distribution and to give low-frequent large-angle scattering. The angular distribution derived by this splitting method agrees well with the distribution derived by the Molière-Bethe theory. Qualitative and quantitative properties of our distribution is investigated by analytical methods.

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2. Evaluation of Molière angular distribution with the splitting cross-section method

As Molière splitting angle $\chi_{\rm B}$ was a little larger for the moderate scattering to produce the central gaussian distribution within the penetration depth, we searched for another smaller splitting angle and have found the well-known characteristic angle $\chi_{\rm C}$ [2], above which the single-scattering occurs once within the penetration depth, plays an adequate splitting angle [3]:

$$\chi_{\rm C} = \theta_{\rm G}/\sqrt{\Omega} \quad \text{with} \quad \theta_{\rm G} = (K/E)\sqrt{t}. \tag{1}$$

Then the Molière angular distribution under the fixed energy process is described by our extended formula [1]:

$$\begin{aligned} 2\pi f(\vartheta) &= f^{(0)}(\vartheta) + \frac{1}{\ln n_{\rm R}}\{f^{(1)}(\vartheta) + f_1^{(1)}(\vartheta)\ln\tau\} \\ &\quad + \frac{1}{(\ln n_{\rm R})^2}\{f^{(2)}(\vartheta) + f_1^{(2)}(\vartheta)\ln\tau + f_2^{(2)}(\vartheta)(\ln\tau)^2\} + \cdots, \end{aligned} \tag{2}$$

with $n_{\rm R}$ of number of single scattering within the depth,

$$n_{\rm R} = t/(\Omega e^{-\Omega+2-2C}), \tag{3}$$

and

$$\vartheta \equiv \theta/(\chi_{\rm C}\sqrt{\ln n_{\rm R}}) \quad \text{and} \quad \tau = e^{2-2C}\ln n_{\rm R}. \tag{4}$$

Taking account the large-angle scattering $\sigma_{\rm L}$ occurs k-times in Poisson probability of $e^{-1}/k!$ within the depth, this distribution (2) is also evaluated as

$$f(\theta)d\vec{\theta} = e^{-1}d\vec{\theta}\sum_{k=0}^{\infty}\frac{1}{k!}N * \sigma_{\rm L}^{(k)}, \tag{5}$$

where $N * \sigma_{\rm L}^{(k)}$ denotes the folding integral between the central distribution $N(\theta)$ and the k-times large-angle scattering $\sigma_{\rm L}^{(k)}$.

3. Evaluation of large-angle double scattering

We describe the normalized single scattering σ_1 as

$$\sigma_1(\rho)d\vec{\rho} = \pi^{-1}\rho^{-4}d\vec{\rho} \quad \text{with} \quad \rho > 1. \tag{6}$$

Then the probability density of the double scattering $\sigma_1^{(2)}$ of the normalized single scattering $\sigma_1^{(1)}$ can be derived as

$$\begin{aligned} \sigma_1^{(2)}(\rho)d\vec{\rho} &= \pi^{-2}d\vec{\rho}\iint_{\rho'>1,|\vec{\rho}-\vec{\rho}'|>1}|\vec{\rho}-\vec{\rho}'|^{-4}\rho'^{-4}d\vec{\rho} \\ &= 4\pi^{-2}d\vec{\rho}\int_1^{\infty}\rho'^{-3}d\rho'\int_{\phi_0}^{\pi}(\rho^2+\rho'^2-2\rho\rho'\cos\phi)^{-2}d\phi, \end{aligned} \tag{7}$$

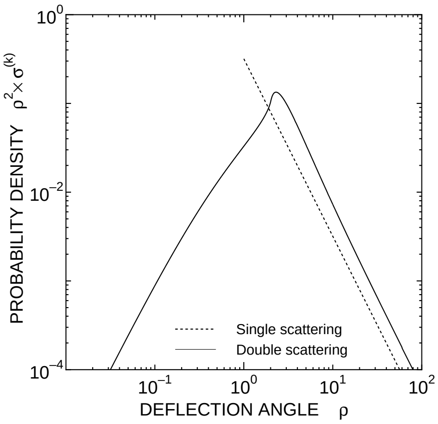

Fig. 1. Probability density of the single and the double scattering.

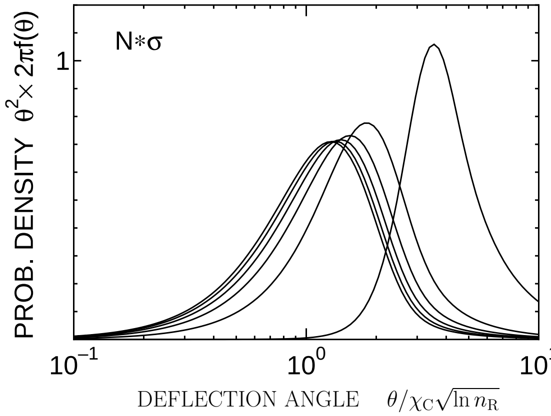

Fig. 2. Normal distribution folded with single scattering. $t = e^k\Omega e^{-\Omega}$ with $k = 1, 2, 3, \cdots, 6$, from right to left.

where $\cos\phi_0 = \mathrm{Min}[1, \rho/2\rho']$. Asymptotic values are effective as

$$\sigma_1^{(2)}(\rho)d\vec{\rho} \simeq (3\pi)^{-1}(1 - 6\rho/\pi + 3\rho^2)d\vec{\rho} \qquad (\rho < 0.06) \tag{8}$$

$$\simeq 2\pi^{-1}\rho^{-4}\{1 + 4\rho^{-2}(2C - 3 + 2\ln\rho)\}d\vec{\rho} \qquad (\rho > 60). \tag{9}$$

The results are indicated in Fig. 1.

4. The single and the double scattering folded with central gaussian distribution

We derive the folding integrals of normal distribution

$$N_{\mathrm{a}}d\vec{\rho} = (\pi a^2)^{-1}e^{-\rho^2/a^2}d\vec{\rho} \tag{10}$$

with the single scattering (6) and the double scattering (7):

$$\begin{aligned} N_{\mathrm{a}} * \sigma_1^{(k)}d\vec{\rho} &= (\pi a^2)^{-1}d\vec{\rho}\iint \sigma^{(k)}(\rho')e^{-(\vec{\rho}-\vec{\rho}')^2/a^2}d\vec{\rho}' \\ &= a^{-2}e^{-\rho^2/a^2}d\vec{\rho}\int_0^\infty I_0(2\rho\rho'/a^2)\sigma^{(k)}(\rho')e^{-\rho'^2/a^2}2\rho' d\rho'. \end{aligned} \tag{11}$$

At $\rho \gg 1$, we have the following asymptotic formulae:

$$N_{\mathrm{a}} * \sigma_1^{(1)}d\vec{\rho} \simeq \pi^{-1}\rho^{-4}d\vec{\rho}(1 + 4a^2\rho^{-2} + 18a^4\rho^{-4} + \cdots), \tag{12}$$

$$\begin{aligned} N_{\mathrm{a}} * \sigma_1^{(2)}d\vec{\rho} \simeq\ & 2\pi^{-1}\rho^{-4}d\vec{\rho}[1 + 4\rho^{-2}(2\ln\rho + a^2 - 1) \\ & + 18a^2\rho^{-4}(4\ln\rho + a^2 - 10/3) + \cdots]. \end{aligned} \tag{13}$$

The results are indicated in Figs. 2 and 3.

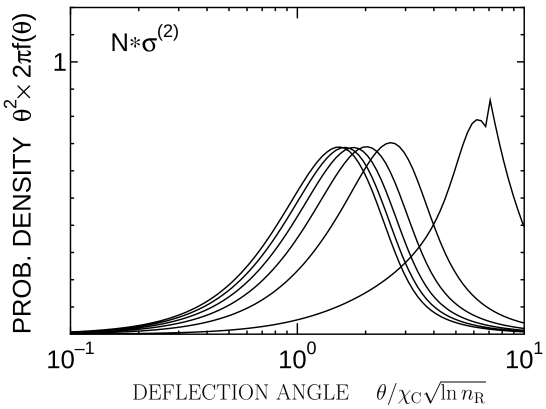

Fig. 3. Normal distribution folded with double scattering. $t = e^k \Omega e^{-\Omega}$ with $k = 1, 2, 3, \cdots, 6$, from right to left.

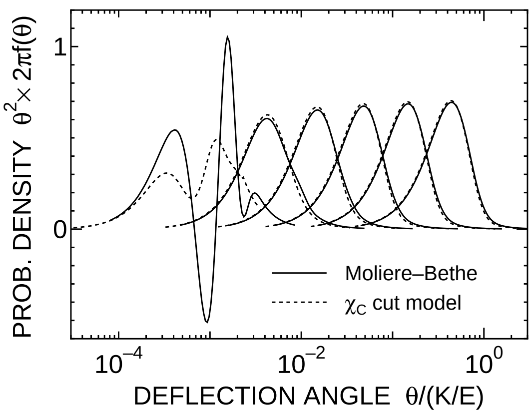

Fig. 4. Molière angular distribution derived through the splitting cross-section method divided at χ_{C}. $t = e^{2k+1}\Omega e^{-\Omega}$ with $k = 0, 1, 2, \cdots, 5$, from left to right.

5. Molière angular distribution derived by splitting cross-section method divided at χ_{C}

With the cross-section divided at χ_{C}, the moderate scattering produces accurate enough gaussian distribution of width $\chi_{\mathrm{C}}\sqrt{\ln n_{\mathrm{R}}}$. So putting $\rho = \theta/\chi_{\mathrm{C}}$ and $a = \sqrt{\ln n_{\mathrm{R}}}$, we can derive the Molière angular distribution by folding k-times large-angle scattering with the central gaussian distribution in Poisson probability as (5). The result evaluated by the first three terms (normalized) is indicated in Fig. 4.

6. Conclusions and discussions

Formulation of the splitting cross-section method for multiple Coulomb scattering process was investigated theoretically with the Poisson probability distribution. Applying the folded angular distribution between the normal distribution produced by the moderate scatterings and the subsequent single and double large-angle scatterings, we have confirmed the normal distribution and the additive large-angle scatterings well reproduce the Molière angular distribution, when we split the cross-section at χ_{C}. The method used here will be valuable for quantitative analyses of other models using splitting cross-section [4,5].

References

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